

$$G = GL_2(\mathbb{Q}_p) \quad \text{Reptors}(G) \quad G \curvearrowright \mathcal{O}_L\text{-mods}$$

$$\text{Ban}^{\text{adm}}(G) \quad G \curvearrowright L\text{-Banach space}$$

$$\mathcal{O}_L[G] \quad G\text{-inv. norm}$$

$$\curvearrowright \Pi^0 \in \Pi$$

$$\Pi^0 / \varpi^n \in \text{Rep}^{\text{tors}}(G)$$

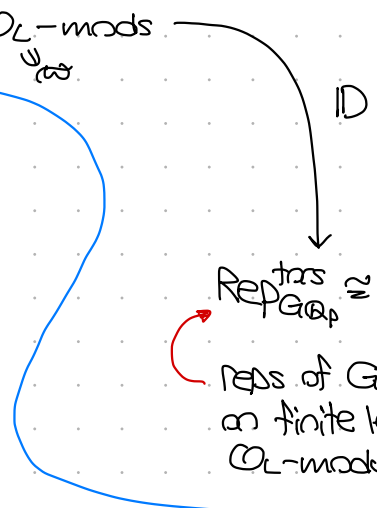
$$\downarrow \\ \text{ID}(\Pi^0 / \varpi^n) \in \mathcal{O}_L^{\text{ét, tors}}$$

$$\text{ID}(\Pi) := L \otimes_{\mathcal{O}_L} \left(\varprojlim_n \text{ID}(\Pi^0 / \varpi^n) \right) \in \text{Rep}_{\mathbb{Q}_p}^{\text{f.d.}}$$

rep. of $G_{\mathbb{Q}_p}$ on L-v.s.

$$\text{Rep}_{\mathbb{Q}_p}^{\text{tors}} \cong \mathcal{O}_L^{\text{ét, tors}}$$

reps of $G_{\mathbb{Q}_p}$ on finite length \mathcal{O}_L -modules



Dotto-Emerton-Gee (geometrization of p-adic LLC)
(categorical p-adic LLC)

Enlarge $\text{Reptors}(G)$ to get enough injectives.

Convenient to fix $\xi: Z \rightarrow \mathcal{O}_L^\times$, $Z := \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} \in G$.

$$\text{Reptors}_{\xi}(G) \subset \text{Reptors}(G)$$

$$\bigcap \text{Reptors}_{\xi}^{\text{f}}(G) = \left\{ \Pi \in \text{Rep}_{\xi}^{\text{sm}}(G) \mid \forall v \in \Pi, \mathcal{O}_L[G] \cdot v \in \text{Reptors}_{\xi}(G) \right\}$$

\uparrow Π has central character ξ , i.e. $z \cdot v = \xi(z)v \quad \forall v \in \Pi$

- ⊛ closed under direct limits
- ⊛ has enough injectives

• $\text{Rep}_{\xi}^{\text{f}}(G)$ satisfies assumptions of a theorem of Gabriel.

Thm: $\text{Rep}_{\xi}^{\text{f}}(G) = \prod_B \text{Rep}_B$

B runs over equivalence classes of irreducibles with equivalence relation generated by: $\Pi_1 \sim \Pi_2$ if \exists non-split extension

in $\text{Rep}_{\xi}^{\text{f}}(G)$.

$$\begin{aligned} & 0 \rightarrow \Pi_2 \rightarrow \tau \rightarrow \Pi_1 \rightarrow 0 \\ \text{or} & 0 \rightarrow \Pi_1 \rightarrow \tau \rightarrow \Pi_2 \rightarrow 0 \end{aligned}$$

$\bigsqcup_B \text{Rep}_B \leftarrow \text{objects are } \bigoplus_B \sigma_B, \sigma_B \in \text{Rep}_B$

Rep_B is $\pi \in \text{Rep}^H(G)$ s.t. all irreducible subquotients lie in B .

We can describe Rep_B as a module category.

Set $\Pi_B := \bigoplus_{\pi \in B/\text{isom.}} \pi \hookrightarrow J_B$ is largest semi-simple sum in J_B , which captures non-semi-simple info.

↑ injective envelope.

$$E_B := \text{End}_{\mathcal{O}_L[G]}(J_B)^{\text{op}}$$

Thm (Gabriel): $\text{Rep}_B \xrightarrow{\text{profinite}} \text{left } \text{End}(J_B)\text{-modules}$

$$\pi \longmapsto \text{Hom}_{\mathcal{O}_L[G]}(\pi, J_B)$$

is an anti-equivalence of categories.

Cor: $\text{Rep}_B \xrightarrow{\text{discrete}} \text{left } E_B\text{-module}$
 $\pi \longmapsto (\text{Hom}_{\mathcal{O}_L[G]}(\pi, J_B))^{\vee}$

is an equivalence of categories.

Examples of blocks B :

① $B = \{\pi\}$, $\pi =$ supersingular rep. of $GL_2(\mathbb{Q}_p)$ on k -v.s (abs. irred.)

e.g. $\frac{\text{c-Ind}_{\mathbb{Z}}^G \text{Sym}^r(k_L^{\oplus 2})}{T-\mathfrak{a}_p} \leftarrow \mathbb{Z}$ acts by $\bar{\rho}: \mathbb{Z} \rightarrow \mathbb{Z}_p^{\times}$
 $x \mapsto (x|x|)^r$

② $B = \{ \text{Ind}_B^G \chi_1 \otimes \chi_2 \omega^{-1}, \text{Ind}_B^G \chi_2 \otimes \chi_1 \omega^{-1} \}$, $\chi_1 \chi_2^{-1} \neq 1, \omega \neq 1$

$$\begin{aligned} \chi_i &: \mathbb{Q}_p^{\times} \rightarrow k_L^{\times} \\ \omega &: \mathbb{Q}_p^{\times} \rightarrow k_L^{\times} \\ x &\mapsto |x| \text{ mod } \mathfrak{a} \end{aligned}$$

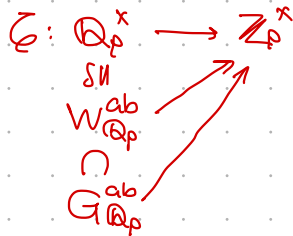
Blocks \longleftrightarrow s.s. 2-dim reps. of $G_{\mathbb{Q}_p}/k_L$.

Colmez: $\text{ID}(\pi) = \text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_2}} \omega_2^{r+1} = \bar{\rho}$

$$\left(\bar{\rho}|_{\mathbb{I}_{\mathbb{Q}_p}} = \omega_2^{r+1} \oplus \omega_2^{r+1+p}, \det \bar{\rho} = \omega^{r+1} \right)$$

$R_{\bar{\rho}}^{\zeta, \epsilon}$ ← on Galois side, normalize by $\bar{\rho}$ -adic cyclotomic character ϵ .

deformation ring for $\bar{\rho}$:
classifies lifts of $\bar{\rho}$ to
reps. on free rank 2
modules over Artin local
 \mathcal{O}_L -algebras with $\det = \zeta \epsilon$.



Thm (Paskunas):

$$E_B \cong R_{\bar{\rho}}^{\zeta, \epsilon} \left(\cong \mathcal{O}_L[[x_1, x_2, x_3]] \right) \quad \begin{array}{l} \dim \mathfrak{m}/(\mathfrak{m}^2, \omega) \\ = \dim H^1(G_{\mathbb{Q}_p}, \text{ad}^{\circ} \bar{\rho}) \end{array}$$

($\Rightarrow E_B$ commutative! only works for s.s. blocks)

(*) $\text{Ext}_{L[G]}^1(\pi_{\text{ss}}, \pi_{\text{ss}}) \cong 3\text{-dim.}$

Moreover, \swarrow Pontryagin dual

$$\underbrace{D(J_B)^{\vee}(\epsilon^{r+1})}_{\substack{\mathcal{O}_L[G_{\mathbb{Q}_p}]\text{-module} \\ E_B\text{-module from} \\ \text{action of } E_B \text{ on } J_B}} \cong \underbrace{\rho^{\text{un}}}_{\substack{\text{universal rep. of } G_{\mathbb{Q}_p} \\ \text{on a free rank 2 } R_{\bar{\rho}}^{\zeta, \epsilon}\text{-mod.} \\ \substack{S \\ E_B}}}.$$

Define a map from:

$$G_{\mathbb{Q}_p}\text{-reps} \longrightarrow GL_2(\mathbb{Q}_p)\text{-reps}$$

Start with $\rho: G_{\mathbb{Q}_p} \rightarrow GL_2(L)$. $\det(\rho) = \zeta \epsilon$

Choose a $G_{\mathbb{Q}_p}$ -stable lattice, up to conjugating, we get

$$\rho: G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathcal{O}_L)$$

Assume $\rho \bmod \omega \cong \bar{\rho}$.

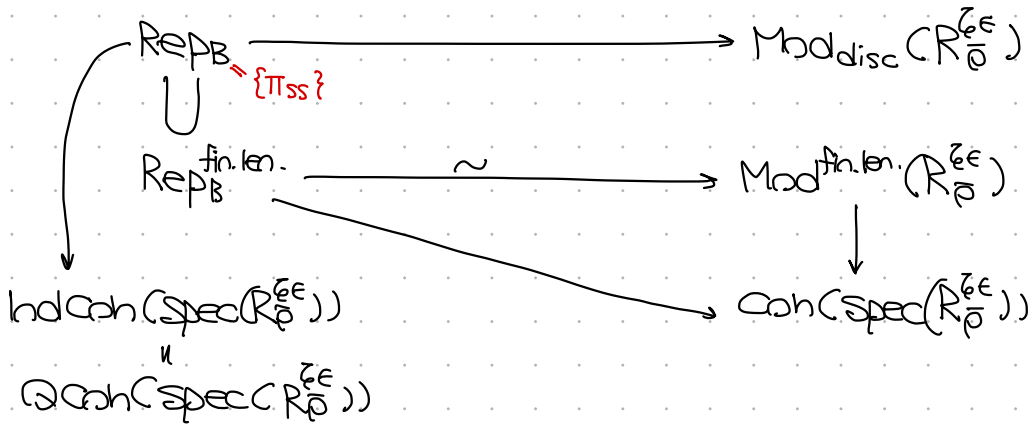
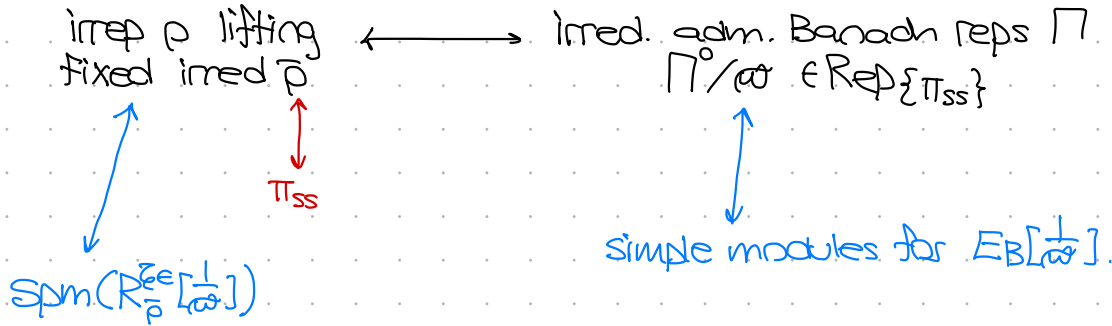
ρ gives map $R_{\bar{\rho}}^{\zeta, \epsilon} \xrightarrow{f} \mathcal{O}_L$ (\mathcal{O}_L -alg map)

$$\rho \cong \rho^{\text{un}} \otimes_{R_{\bar{\rho}}^{\zeta, \epsilon}} \mathcal{O}_L \cong \mathcal{O}_L \otimes_{f, E_B} D(J_B)^{\vee}(\epsilon^{r+1}) = D(J_B \otimes_{E_B, f} \mathcal{O}_L)^{\vee}(\epsilon^{r+1})$$

$$\text{Coh} := \text{Set } \pi(\rho) := J_B \otimes_{E_B, f} \mathcal{O}_L$$

$$\rho \simeq \text{ID}(\pi(\rho))^\vee(\epsilon^{r+1})$$

Can be used to establish bijection.



Other blocks, also get:

