Paskunas's work on p-adic LLC

7 Mar

$G = GL_2(\mathbb{Q}_p)$ $\quad \mathrm{Rep}_{\mathrm{tors}}(G)$ $\quad G \curvearrowright \mathcal{O}_L\text{-mods}$ (killed by ϖ^n)

$\mathbb{D}$

$\mathrm{Ban}^{\mathrm{adm}}(G)$ $\quad G \curvearrowright L$-Banach space.

$\mathcal{O}_L[G]$ $\quad$ G-inv. norm

$\Pi^\circ \subseteq \Pi$

$\Pi^\circ/\varpi^n \in \mathrm{Rep}^{\mathrm{tors}}(G)$

$\downarrow$

$\mathbb{D}(\Pi^\circ/\varpi^n) \in \varphi\Gamma^{\text{ét},\mathrm{tors}}_{\mathcal{O}_\mathcal{E}}$

$\mathrm{Rep}^{\mathrm{tors}}_{G_{\mathbb{Q}_p}} \simeq \varphi\Gamma^{\text{ét},\mathrm{tors}}_{\mathcal{O}_\mathcal{E}}$

reps of $G_{\mathbb{Q}_p}$ on finite length $\mathcal{O}_L$-modules

$$\mathbb{D}(\Pi) := L \otimes_{\mathcal{O}_L} \left(\varprojlim_n \mathbb{D}(\Pi^\circ/\varpi^n) \right) \in \mathrm{Rep}^{\mathrm{f.d.}}_{G_{\mathbb{Q}_p}}$$

rep. of $G_{\mathbb{Q}_p}$ on L-v.s.

Dotto-Emerton-Gee (geometrization of p-adic LLC) (categorical p-adic LLC)

Enlarge $\mathrm{Rep}_{\mathrm{tors}}(G)$ to get enough injectives.

Convenient to fix $\zeta : Z \longrightarrow \mathcal{O}_L^\times$, $\quad Z := \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} \in G$.

$$\mathrm{Rep}_{\mathrm{tors},\zeta}(G) \subset \mathrm{Rep}_{\mathrm{tors}}(G)$$

π has central character ζ, i.e. $z\cdot v = \zeta(z)v \quad \forall v \in \Pi$

$$\mathrm{Rep}^{\mathrm{lf}}_\zeta(G) = \left\{ \pi \in \mathrm{Rep}^{\mathrm{sm}}_\zeta(G) \;\middle|\; \forall v \in \pi,\ \mathcal{O}_L[G]\cdot v \in \mathrm{Rep}_{\mathrm{tors},\zeta}(G) \right\}$$

* closed under direct limits
* has enough injectives

- $\mathrm{Rep}^{\mathrm{lf}}_\zeta(G)$ satisfies assumptions of a theorem of Gabriel.

Thm: $\mathrm{Rep}^{\mathrm{lf}}_\zeta(G) = \prod_B \mathrm{Rep}_B$

B runs over equivalence classes of irreducibles with equivalence relation generated by: $\pi_1 \sim \pi_2$ if $\exists$ non-split extension

$$0 \longrightarrow \pi_2 \longrightarrow \tau \longrightarrow \pi_1 \longrightarrow 0$$

in $\mathrm{Rep}^{\mathrm{lf}}_\zeta(G)$. $\quad$ or $\quad 0 \longrightarrow \pi_1 \longrightarrow \tau \longrightarrow \pi_2 \longrightarrow 0$

$\prod_B \mathrm{Rep}_B \longleftarrow$ objects are $\bigoplus_B \sigma_B$, $\sigma_B \in \mathrm{Rep}_B$

Rep_B is $\pi \in \mathrm{Rep}^{lf}(G)$ s.t. all irreducible subquotients lie in B.

We can describe Rep_B as a module category.

Set $\pi_B := \bigoplus_{\pi \in B/\mathrm{isom.}} \pi \xhookrightarrow{\quad} J_B$

injective envelope. π_B is largest semi-simple sum in J_B, which captures non-semi-simple info.

$E_B := \mathrm{End}_{\mathcal{O}[G]}(J_B)^{op}$.

Thm (Gabriel): $\mathrm{Rep}_B \longrightarrow$ left profinite $\mathrm{End}(J_B)$-modules

$$\pi \longmapsto \mathrm{Hom}_{\mathcal{O}[G]}(\pi, J_B)$$

is an anti-equivalence of categories.

Cor: $\mathrm{Rep}_B \longrightarrow$ left discrete E_B-module

$$\pi \longmapsto (\mathrm{Hom}_{\mathcal{O}[G]}(\pi, J_B))^\vee$$

is an equivalence of categories.

Examples of blocks B:

① $\mathcal{B} = \{\pi\}$, π = supersingular rep. of $GL_2(\mathbb{Q}_p)$ on k-v.s (abs. irred.)

e.g. $\dfrac{c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r(k_L^{\oplus 2})}{T - a_p} \longleftarrow Z$ acts by $\zeta : Z \longrightarrow \mathbb{Z}_p^\times$, $x \longmapsto (x|x|)^r$

② $\mathcal{B} = \{ \mathrm{Ind}_B^G \chi_1 \otimes \chi_2 \omega^{-1}, \mathrm{Ind}_B^G \chi_2 \otimes \chi_1 \omega^{-1} \}$, $\chi_1 \chi_2^{-1} \neq 1, \omega^{\pm 1}$

$$\chi_i : \mathbb{Q}_p^\times \longrightarrow k_L^\times$$
$$\omega : \mathbb{Q}_p^\times \longrightarrow k_L^\times$$
$$x \longmapsto x|x| \bmod \varpi$$

$$\boxed{\text{Blocks} \longleftrightarrow \text{s.s. 2-dim reps. of } G_{\mathbb{Q}_p}/k_L.}$$

Colmez: $\mathbb{D}(\pi) = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1} = \bar{\rho}$

$$\left(\bar{\rho}|_{I_{\mathbb{Q}_p}} = \omega_2^{r+1} \oplus \omega_2^{(r+1)p}, \quad \det \bar{\rho} = \omega^{r+1} \right)$$

$R_{\bar\rho}^{\zeta\cdot\epsilon}$ ⟵ on Galois side, normalize by p-adic cyclotomic character ϵ

deformation ring for $\bar\rho$: classifies lifts of $\bar\rho$ to reps on free rank 2 modules over Artin local $\mathcal{O}_L$-algebras with $\det = \zeta\epsilon$.

$$\begin{tikzcd} \zeta : \mathbb{Q}_p^\times \arrow[r] \arrow[d, phantom, "\wr\Vert"] & \mathbb{Z}_p^\times \\ W_{\mathbb{Q}_p}^{ab} \arrow[ur] \arrow[d, phantom, "\cap"] & \\ G_{\mathbb{Q}_p}^{ab} \arrow[uur] & \end{tikzcd}$$

Thm (Paskunas):

$$E_B \cong R_{\bar\rho}^{\zeta\epsilon} \left(\cong \mathcal{O}_L[[x_1, x_2, x_3]]\right)$$

$\dim \mathfrak{m}/(\mathfrak{m}^2, \varpi)$
$= \dim H^1(G_{\mathbb{Q}_p}, \mathrm{ad}^0 \bar\rho)$

($\Rightarrow E_B$ commutative! Only works for s.s. blocks)

⊛ $\mathrm{Ext}^1_{k_L[G], \zeta}(\pi_{ss}, \pi_{ss})$ 3-dim.

Moreover,

Pontryagin dual

$$\underbrace{D(J_B)^\vee}(\epsilon^{r+1}) \cong \rho^{un}$$

$\mathcal{O}_L[G_{\mathbb{Q}_p}]$-module
E_B-module from action of E_B on J_B.

universal rep. of $G_{\mathbb{Q}_p}$ on a free rank 2 $R_{\bar\rho}^{\zeta\epsilon}$-mod. ($R_{\bar\rho}^{\zeta\epsilon} \cong E_B$)

Define a map from:

$$G_{\mathbb{Q}_p}\text{-reps} \longrightarrow GL_2(\mathbb{Q}_p)\text{-reps}$$

Start with $\rho : G_{\mathbb{Q}_p} \longrightarrow GL_2(L)$. $\det(\rho) = \zeta\epsilon$

Choose a $G_{\mathbb{Q}_p}$-stable lattice, up to conjugating, we get

$$\rho : G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathcal{O}_L)$$

Assume $\rho \bmod \varpi \cong \bar\rho$.

ρ gives map $R_{\bar\rho}^{\zeta\epsilon} \xrightarrow{f} \mathcal{O}_L$ ($\mathcal{O}_L$-alg map)

$$\rho \cong \rho^{un} \otimes_{R_{\bar\rho}^{\zeta\epsilon}} \mathcal{O}_L \cong \mathcal{O}_L \otimes_{f, E_B} D(J_B)^\vee(\epsilon^{r+1}) = D(J_B \otimes_{E_B, f} \mathcal{O}_L)^\vee(\epsilon^{r+1})$$

Cor: Set $\Pi(\rho)^\circ := J_B \otimes_{E_B, f} \mathcal{O}_L$

$$\rho \cong \mathbb{D}(\Pi(\rho))^\vee(\epsilon^{r+1})$$

Can be used to establish bijection.

$$\begin{tikzcd}
\begin{array}{c}\text{irrep } \rho \text{ lifting} \\ \text{fixed irred } \bar\rho\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{c}\text{Irred. adm. Banach reps } \Pi \\ \Pi^\circ/\varpi \in \mathrm{Rep}_{\{\pi_{ss}\}}\end{array} \arrow[d, leftrightarrow] \\
\mathrm{Spm}(R^{\zeta\epsilon}_{\bar\rho}[\tfrac{1}{\varpi}]) & \text{simple modules for } E_B[\tfrac{1}{\varpi}].
\end{tikzcd}$$

$$\bar\rho \longleftrightarrow \pi_{ss}$$

$$\begin{tikzcd}
\mathrm{Rep}_B = \{\pi_{ss}\} \arrow[rr] \arrow[dd, bend right] & & \mathrm{Mod}_{disc}(R^{\zeta\epsilon}_{\bar\rho}) \\
\mathrm{Rep}_B^{fin.len.} \arrow[u, phantom, "\cup"] \arrow[rr, "\sim"] \arrow[drr] & & \mathrm{Mod}^{fin.len.}(R^{\zeta\epsilon}_{\bar\rho}) \arrow[d] \\
\mathrm{IndCoh}(\mathrm{Spec}(R^{\zeta\epsilon}_{\bar\rho})) & & \mathrm{Coh}(\mathrm{Spec}(R^{\zeta\epsilon}_{\bar\rho}))
\end{tikzcd}$$

$$\mathrm{IndCoh}(\mathrm{Spec}(R^{\zeta\epsilon}_{\bar\rho})) = \mathrm{QCoh}(\mathrm{Spec}(R^{\zeta\epsilon}_{\bar\rho}))$$

Other blocks, also get:

$$\begin{tikzcd}
\mathrm{Rep}_B \arrow[r] \arrow[d, "\text{restrict to } B"'] & \underbrace{\mathrm{IndCoh}}_{\text{derived}}(\mathfrak{X}_{\bar\rho}) \arrow[d, "\text{localize + complete at a point} \leftrightarrow \bar\rho"] \\
D(\mathrm{Rep}_\zeta^{l\text{-adm}}(G)) \arrow[r] & \mathrm{IndCoh}(\mathfrak{X}^{EG})
\end{tikzcd}$$

stacks of Galois reps (Wang-Erickson)

N. + C. Johanssen
+ C. Wang Erickson.

DEG
EGH