Properties of the Montréal functor

7 Mar

$G = GL_2(\mathbb{Q}_p)$. $L/\mathbb{Q}_p$ fin. ext with uniformizer ϖ.

$\Gamma := \mathbb{Z}_p^\times$. $\mathcal{O}_\mathcal{E}^+ := \mathcal{O}_L[\![T]\!]$, $\mathcal{O}_\mathcal{E} := \left\{ \sum_{k\in\mathbb{Z}} a_k T^k \;\middle|\; a_k \to 0,\ k \to -\infty \right\}$

$$\Gamma, \varphi \text{ by } \begin{array}{l} \sigma^a : T \mapsto (1+T)^a - 1 \\ \varphi : T \mapsto (1+T)^p - 1 \end{array}$$

A <u>(φ,Γ)-module</u> M is a top. $\mathcal{O}_\mathcal{E}$ (or $\mathcal{O}_\mathcal{E}^+$)-module with cts actions of φ_M, σ_M^a st.

- actions commute w/ σ^a, φ.
- $\Gamma \times M \longrightarrow M$ cts.

$$\mathrm{Rep}_{\mathrm{tors}}(G) := \{ \text{smooth } \mathcal{O}_L[G]\text{-reps} + \text{finiteness conditions} \}$$

$\ni \pi$ e.g. π^K is finite length over $\mathcal{O}_L \Rightarrow$ finite set.

<u>Montreal functor</u>: $\mathbb{D} : \mathrm{Rep}_{\mathrm{tors}}(G) \longrightarrow (\varphi,\Gamma)\text{-Mod}$

"informal construction"

$$\mathcal{O}_L[\![T]\!] \longleftrightarrow \mathcal{O}_L\left[\!\left[\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}\right]\!\right]$$

$$T+1 \longleftrightarrow \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

$$\text{actions of } \varphi, \sigma^a \longleftrightarrow \left\{ \begin{array}{l} \varphi \longrightarrow \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & px \\ & 1 \end{pmatrix} \\ \sigma^a \longrightarrow \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix} \\ \varphi_M \longrightarrow \text{mult by } \begin{pmatrix} p & \\ & 1 \end{pmatrix} \end{array} \right.$$

$\rightsquigarrow$ generate action of $P^+ := \begin{pmatrix} \mathbb{Z}_p\setminus\{0\} & \mathbb{Z}_p \\ & 1 \end{pmatrix}$

In general, $\pi \rightsquigarrow \mathcal{W}(\pi) =$ "nice submods"
$\cup$
$\mathcal{W}^0(\pi) =$ "std presentation, ..."

$$\mathcal{W}(\pi) \ni W \rightsquigarrow D_W^\natural(\pi)$$
$$\cup$$
$$\mathcal{W}^0(\pi) \ni W^0 \rightsquigarrow D_{W^0}^+(\pi) \in (\varphi,\Gamma)\text{-Mod}(\mathcal{O}_\mathcal{E}^+)$$

$$\mathbb{D}(\pi) := \varprojlim_{W^0} D_{W^0}^+(\pi) \otimes_{\mathcal{O}_\mathcal{E}^+} \mathcal{O}_\mathcal{E}$$

$$D_W^\natural(\pi) := \Phi(I_{\mathbb{Z}_p}(W))^\vee \qquad I_{\mathbb{Z}_p}(W) = \left\{ \sum \begin{pmatrix} p^n & a \\ & 1 \end{pmatrix} W \;\middle|\; a + p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p,\ a \in \mathbb{Q}_p \right\}$$

Today: $\mathbb{D}(\pi)$ is f.g. and étale.

§ étaleness $\quad R = \mathcal{O}_\mathcal{E}, \mathcal{O}_\mathcal{E}^+$

Def: $M \in (\varphi,\Gamma)\text{-Mod}(R)$ is étale if the map

$$\mathrm{Id}_R \otimes \varphi_M : R \otimes_\varphi M \longrightarrow M \quad \text{is an iso.}$$

$$r \otimes m \longmapsto r \otimes \varphi_M(m)$$

Lemma: $\psi : \bigoplus_{i=0}^{p-1} \pi^\vee \longrightarrow \pi^\vee$

$$(\mu_0, \dots, \mu_{p-1}) \longmapsto \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ & 1 \end{pmatrix} \mu_i$$

restricts to an injective map w/ finite cokernel on $D_W^+(\pi)$

on $(D_W^+(\pi))^{\oplus p} \longrightarrow D_W^+(\pi)$

Pf (sketch): Injectivity: Argue that $D_W^+(\pi)$ is generated over $\mathcal{O}_L$ by elements of the form $\mathbb{1}_{\begin{pmatrix} p^n & a \\ & 1 \end{pmatrix} W} =: \mu_{n,a}$

- $\mathrm{supp}\left(\begin{pmatrix} p & i \\ & 1 \end{pmatrix} \mu_{n,a}\right)$ are disjoint

Almost surjectivity: $\mathrm{coker}\, \psi \hookrightarrow \left\{ \mu \in \pi^\vee \,\middle|\, \mu|_W = 0 = \mu_{\begin{pmatrix} p & i \\ & 1 \end{pmatrix} W} \right\}$

dual to $W + \mathbb{Z} \begin{pmatrix} p & i \\ & 1 \end{pmatrix} W$

Thm: $\mathbb{D}(\pi)$ is étale and exact

Pf: $(\mu_i) \overset{\psi}{\longmapsto} \sum (1+T)^i \varphi(\mu_i)$ with finite cokernel

$\begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}$ $\quad \begin{pmatrix} p & \\ & 1 \end{pmatrix}$

$$\Longrightarrow \mathrm{span}\left(\varphi(D_W^+(\pi))\right) \underset{\substack{\text{finite} \\ \text{index}}}{\subset} D_W^+(\pi)$$

$\rightsquigarrow \otimes \mathcal{O}_\mathcal{E}$ is iso.

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§ Finite generation of $\mathbb{D}(\pi)$

- Colmez "ad hoc"
- Emerton "algebra-based"

Key observation: M being f.g. $\Leftrightarrow$ $M^\vee$ is "admissible"

- A DVR, uniformizer t, $k = A/tA$.
- $F \in \mathrm{End}(A)$ local (i.e. $F(t) \in tA$) s.t. $F \in \mathrm{End}(k)$ is trivial.

(for us, $A = k_\mathcal{E}^+ := \mathcal{O}_L[[t]]/\varpi$. π s.t. $\pi = \pi[\varpi]$, $F = \varphi$) killed by ϖ

- $M \in A\text{-Mod} \rightsquigarrow M[t] := \{ m \in M \mid tm = 0 \}$.
- M is <u>admissible</u> if $M[t]$ is f.d. $/k$ + M is A-torsion.

<u>Thm</u>: The map $M \longmapsto \mathrm{Hom}(M, \mathrm{Frac}(A)/A) =: M^\vee$ (Gabriel '62)
is an anti-equiv. of categories,

$$\{\text{admissible } A\text{-modules}\} \longleftrightarrow \{\text{f.g. } \hat{A}\text{-modules}\}$$

t-adic completion of A

<u>Setup</u>: $W \in \mathcal{W}(\pi)$, goal: show $\Phi(I_{\mathbb{Z}_p}(W))^\vee$ f.g. $\Leftrightarrow$ $\Phi(I_{\mathbb{Z}_p}(W))$ adm.

<u>Step ①</u>: Reduce to $\pi = \pi[\varpi]$ (by passing to quotients $\pi[\varpi^k]/\pi[\varpi^{k-1}]$

<u>Notation</u>: $M(\pi, W) :=$ k_ε^+-submod. of π generated by W.

$$0 \longrightarrow M(\pi,W) \longrightarrow \pi \longrightarrow \pi/M(\pi,W) \longrightarrow 0$$

tor exact sequence

k[F]-torsion b/c W good

$$\begin{tikzcd}
0 \arrow[r] & M(\pi,W)[t] \arrow[r] \arrow[d] & \pi[t] \arrow[r] \arrow[d] & (\pi/M(\pi,W))[t] \arrow[d] \arrow[dll, out=0, in=180] & \\
& M(\pi,W)/t \arrow[r] & \pi/t \arrow[r] & (\pi/M(\pi,W))/t \arrow[r] & 0
\end{tikzcd}$$

(snake lemma)

<u>Lemma</u>: M is admissible $\Leftrightarrow$ M/tM is $k[F]$-torsion

<u>Claim</u>: π/t is $k[F]$-torsion

① $\pi/t \cong H^1_{cts}(N^0, \pi)$ $\qquad$ F acts trivially.

$\qquad\qquad \| $

$\qquad \mathcal{O}_L[[\begin{smallmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{smallmatrix}]]$

② This is f.g. $/k$ by using $(*)$

This finishes proof of: $\boxed{\underline{\text{Thm}}: \ D(\pi) \text{ is f.g.}}$