Properties of the Montreal functor

These notes are quite informal, so I hope I don't offend anyone.

1 Reminder

Here's a brief recap of last week:

- $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $B = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$,
- $L/\mathbb{Q}_p$ finite extension with uniformiser ϖ,
- $\Gamma := \mathbb{Z}_p^\times$,
- $\mathcal{O}_\mathcal{E} := \mathcal{O}_L[[T]]$, $\mathcal{O}_\mathcal{E}^+ = \left\{\sum_{k \gg -\infty} a_k T^k\right\}$
- action $\Gamma, \varphi \circlearrowleft \mathcal{O}_\mathcal{E}$ by $\sigma^a : T \mapsto (1+T)^a - 1$ and $\varphi(T) = (1+T)^p - 1$.
- a *(φ, Γ)-module* is a topological $\mathcal{O}_\mathcal{E}$ or $\mathcal{O}_\mathcal{E}^+$-module M with continuous lifts of σ^a and φ denoted σ_M^a and φ_M such that $\Gamma \times M \to M$ is continuous.
- We defined the *Montreal functor* $\boldsymbol{D}$ from the category $\mathrm{Rep}_{\mathrm{tors}}(G)$ consisting of smooth $\mathcal{O}_L[G]$-representations Π with finiteness conditions (in part. Π^K finite set for all $K \subset G$ compact open)
- For a subclass of representations, we defined this by identifying $\mathcal{O}_L[[T]]$ with $\mathcal{O}_L\left[\left[\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)\right]\right]$, and matching up φ, σ^a with elements of $P^+ = \left(\begin{smallmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ as follows:

$$\begin{aligned} \varphi &\rightsquigarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & px \\ 0 & 1 \end{pmatrix} \\ \sigma^a &\rightsquigarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \\ \varphi_M &\rightsquigarrow \begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{1}$$

In the general case, for a given Π we defined sets $\mathcal{W}(\Pi)$ of "nice submodules" which were just the right size, and a set of submodules $\mathcal{W}^0(\Pi) \subset \mathcal{W}(\Pi)$ of modules "which gave standard presentation". For $W \in \mathcal{W}(\Pi)$, this gave rise to $\boldsymbol{D}_W^\natural(\Pi) \in (\varphi, \Gamma)\text{-}\mathrm{Mod}(\mathcal{O}_\mathcal{E}^+)$, which were all isomorphic for different W. Taking a limit gave $\boldsymbol{D}(\Pi) = \lim_W \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} \boldsymbol{D}_W^\natural(\Pi)$ which was independent of choice. Furthermore, for $W \in \mathcal{W}^0(\Pi)$, we also defined another functor $\boldsymbol{D}_W^+(\Pi)$ which had finite index in $\boldsymbol{D}_W^\natural(\Pi)$.

Explicitly, this was given by $\Phi\left(I_{\mathbb{Z}_p}(W)\right)^\vee$, which is something like

$$\Phi\left(I_{\mathbb{Z}_p}(W)\right) = \sum \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} W, \quad \text{for } a + p^n\mathbb{Z}_p \subset \mathbb{Z}_p, \quad a \in \mathbb{Q}_p. \tag{2}$$

2 Étaleness of $\boldsymbol{D}(\Pi)$

This is not so hard: recall that a (φ, Γ)-module M is *étale* if the map

$$\mathrm{Id}_R \otimes \varphi_M : \mathcal{O}_\mathcal{E} \otimes_\varphi M \to M \tag{3}$$

is an isomorphism. Since the map is given by $(r, m) \mapsto r \cdot \varphi_M(m)$, this is equivalent to saying that $\varphi_M(M)$ generates M over R.

As in the previous talk, we will use the standard presentation, $D_W^+(\Pi)$ then tensor up with $\mathcal{O}_{\mathcal{E}}$.

Lemma 2.1: Let $\psi : \oplus_{i=0}^{p-1} \Pi^\vee \to \Pi^\vee$ be the map defined by

$$\psi(\mu_0, ..., \mu_{p-1}) = \sum_i \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \mu_i. \tag{4}$$

Then $\psi|_{D_W^+(\Pi)}$ is injective, and its image has finite index in $D_W^+(\Pi)$.

Proof (sketch): Note that the restriction lands in $D_W^+(\Pi)$ since it is closed under P^+. For injectivity, argue that $D_W^+(\Pi)$ is generated by elements μ_i of the form $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right) W$ and for these the summands have disjoint support. So if the image is zero, we must have $\mu_i = 0$ for all i. For almost surjectivity, prove that

$$\operatorname{coker} \psi \subset \left\{ \mu \in \Pi^\vee : \mu|_W = 0 = \mu|_{\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) W}, \forall i \right\}, \tag{5}$$

and this is dual to $W + \sum \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} W$, which is finite length over $\mathcal{O}_L$ because W is, hence a finite set by what James said last week. □

Now we pass to $\mathcal{O}_{\mathcal{E}}^+$: the map becomes $(\mu_i) \mapsto \sum (1+T)^i \varphi(\mu_i)$, which has finite cokernel. Tensoring with $\mathcal{O}_{\mathcal{E}}$ kills the cokernel, and so we get that $\varphi(\boldsymbol{D}(\Pi))$ generates $\boldsymbol{D}(\Pi)$.

Another thing I should state (but won't prove) is:

Theorem 2.2: *The functor* $\boldsymbol{D}$ *is exact.*

3 Finite generation of $\boldsymbol{D}(\Pi)$

This is a lot more effort. Colmez originally used an ad-hoc argument using the classification of supercuspidals of $\mathrm{GL}_2(\mathbb{Q}_p)$, but Emerton swooped in and gave a slightly more conceptual proof. We will follow his version of the proof.

The starting point in this section is the observation that $\boldsymbol{D}(\Pi)$ being finitely generated is (Pontryagin) dual to being "admissible".

Setup: Let A be a DVR with uniformiser t and residue field $k := \frac{A}{t}A$. Suppose A is equipped with a local endomorphism F which reduces to the identity on k. If M is an A-module, let $M[t]$ be the elements of m killed by t. This admits an action of F by $F \cdot m = \frac{F(t)}{t} \cdot m$ which makes sense because $F(t) \in tA$.

Definition 3.1: M is *admissible* if M is A-torsion and $M[t]$ is finite-dimensional over k.

Proposition 3.2: The map $M \mapsto \operatorname{Hom}(M, \operatorname{Frac}(A)/A) =: M^\vee$ is an (anti)equivalence of categorie,

$$\{\text{admissible } A\text{-modules}\} \to \left\{ \text{F.g. } \hat{A}\text{-modules} \right\}, \tag{6}$$

where $\hat{A}$ is the t-adic completion of A.

For a proof of this, see the masters thesis - it is not in Emerton's paper.

We want to apply this to $M = \Phi\Big(I_{\mathbb{Z}_p}(W)\Big)$ as an $A := k_{\mathcal{E}}^{+} := \mathcal{O}_L[[t]]/\varpi$-module, since $\boldsymbol{D}_W^{\natural}(\Pi) = M^{\vee}$ by definition.

A first reduction: we know that $\Pi = \Pi[\varpi^n]$ for some $n \in \mathbb{N}$, and we can assume $n = 1$ by an argument passing to the successive quotients $\Pi[\varpi^k]/\Pi[\varpi^{k-1}]$.

Some notation: let $M(\Pi, W)$ be the submodule of Π generated by W. So $\Phi\Big(I_{\mathbb{Z}_p}(W)\Big) = M(\Pi, W)$.

Lemma 3.3: M is admissible if and only if M/tM is $k[F]$-torsion.

Next, we apply this to the Tor-long exact sequence associated to

$$0 \to M(\Pi, W) \to \Pi \to \Pi/M(\Pi, W) \to 0, \tag{7}$$

(with respect to $\cdot \otimes k$). Let's recall how this works; over a DVR, a short exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{8}$$

gives rise to a long exact sequence

$$0 \to M'[t] \to M[t] \to M''[t] \to M'/tM' \to M/tM \to M''/tM'' \to 0. \tag{9}$$

This reduces the question to showing that $\left(\frac{\Pi}{M(\Pi,W)}\right)[t]$ and $\Pi/t\Pi$ are $k[F]$-torsion. (Cf. p.4 of Emerton's notes.) The first follows from our choice of W, I think, the point being that W generates Π over $\mathcal{O}_L[G]$. A computation using the projective resolution $0 \to k[[t]] \to k[[t]] \to k \to 0$ shows that $\Pi/t\Pi \cong H^1(N_0, \Pi)$, and this is shown to be invariant under F and finitely generated over k, proving the second claim.

Now we have proved (or sketched):

Proposition 3.4: Suppose $\Pi = \Pi[\varpi]$. Then $\Phi\Big(I_{\mathbb{Z}_p}(W)\Big)$ is admissible, hence $\boldsymbol{D}_W^{\natural}(\Pi)$ is finitely generated.

By the aforementioned reduction, this implies:

Theorem 3.5: *The* (φ, Γ)*-module* $\boldsymbol{D}(\Pi)$ *is étale and finitely generated.*