Definition of the Montreal functor (James Taylor) 29 Feb

$L/\mathbb{Q}_p$ finite. $\mathcal{O}_L \ni \varpi$, $G = GL_2(\mathbb{Q}_p)$.

Aim: Define a functor $D : \mathrm{Rep}_{\mathrm{tors}}(G) \longrightarrow \Phi\Gamma(\mathcal{O}_\mathcal{E})$

$$\begin{tikzcd}
\mathrm{Rep}_{\mathrm{tors}}(G) \arrow[r] \arrow[dr, dashed, "\text{(next time)}"'] & \Phi\Gamma(\mathcal{O}_\mathcal{E}) \\
& \Phi\Gamma^{ét}_{\mathrm{tors}, \mathrm{c.o.}}(\mathcal{O}_\mathcal{E}) \arrow[u, hook]
\end{tikzcd}$$

§1 The categories

§1.1: $\mathrm{Rep}_{\mathrm{tors}}(G)$ is the full subcategory of $\mathcal{O}_L[G]$-mod. consisting of objects π s.t.:
- (i) π is smooth, π^K finite length as $\mathcal{O}_L$-mod. $\forall K \underset{c.o.}{\leq} G$.
- (ii) π finite length over $\mathcal{O}_L[G]$.
- (iii) π admits central character.

Rmk: $M \in \mathcal{O}_L$-mod. is finite length $\Leftrightarrow$ f. gen. + torsion
$\Leftrightarrow$ finite as a set.

Rmk: (i) can be replaced by
(i)* π is smooth admissible + $\mathcal{O}_L$-torsion.

§1.2: (φ, Γ)-modules

$$\mathcal{O}_\mathcal{E}^+ := \mathcal{O}_L[\![T]\!] \hookrightarrow \mathcal{O}_\mathcal{E} := \left\{ \sum_{k \in \mathbb{Z}} a_k T^k \;\middle|\; \begin{array}{l} a_k \in \mathcal{O}_L \\ a_k \longrightarrow 0 \text{ as } k \to -\infty \end{array} \right\}$$

- $\mathcal{O}_\mathcal{E}$ is a DVR with uniformizer ϖ.
- $\mathcal{O}_\mathcal{E}$ has coarsest topology s.t. $\mathcal{O}_\mathcal{E} \longrightarrow \mathcal{O}_\mathcal{E}/\varpi = k_L((T))$ is cts. (v_T-topology)

Let $\Gamma := \mathbb{Z}_p^\times$. Let $R \in \{\mathcal{O}_\mathcal{E}^+, \mathcal{O}_\mathcal{E}\}$.

$\mathcal{O}_L$-linear cts. maps: $\varphi : R \to R$, $\varphi(f(T)) := f((1+T)^p - 1)$

$\forall a \in \Gamma$, $\sigma^a : R \to R$, $\sigma^a(f(T)) := f((1+T)^a - 1)$

Def: Let $R \in \{\mathcal{O}_\mathcal{E}^+, \mathcal{O}_\mathcal{E}\}$. Then a (φ, Γ)-module is a triple (M, σ_M, φ_M) s.t.
- M is a top. R-module.
- $\varphi_M : M \longrightarrow M$ cts φ-semilinear
- $\forall a \in \Gamma$, $\sigma_M^a : M \longrightarrow M$.

such that:
- $\Gamma \times M \to M$ is cts, group action
- actions of φ_M, σ_M commute.

Called:
- étale if $\langle \varphi_M(M) \rangle_M = M$
- torsion if M is finite length as R-module.

Lemma: $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} - : \phi\Gamma(\mathcal{O}_{\mathcal{E}}^+) \longrightarrow \phi\Gamma(\mathcal{O}_{\mathcal{E}})$ is <u>exact</u>, and

$\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} M = 0$ whenever M is finite as a set (finite length as $\mathcal{O}_L$-mod.)

$\mathcal{O}_{\mathcal{E}}$ is flat over $\mathcal{O}_{\mathcal{E}}^+$ (as abstract rings w/o topology)

(φ,Γ)-structure

$\mathcal{O}_{\mathcal{E}}^+$-module structure

§2: Getting a (φ,Γ)-module from a representation

Def: $N_0 := \begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix} \subset \begin{pmatrix} \mathbb{Z}_p\setminus\{0\} & \mathbb{Z}_p \\ & 1 \end{pmatrix} =: P^+ \subset G$

- Suppose M is complete topological $\mathcal{O}_L$-module, with $\mathcal{O}_L$-linear action of N_0, s.t. $N_0 \times M \to M$ is cts.

Then the $\mathcal{O}_L[N_0]$-module structure extends to a $\mathcal{O}_L[\![N_0]\!]$-module structure, where

$$\mathcal{O}_L[\![N_0]\!] := \varprojlim_{\substack{U \leq N_0 \\ \text{c.o.}}} \mathcal{O}_L[N_0/U] \hookleftarrow \mathcal{O}_L[N_0]$$

There is an $\mathcal{O}_L$-algebra isomorphism:

$$\mathcal{O}_L[\![N_0]\!] \xrightarrow{\sim} \mathcal{O}_L[\![T]\!]$$
$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \longmapsto 1+T.$$

$\rightsquigarrow$ view M as a $\mathcal{O}_{\mathcal{E}}^+ := \mathcal{O}_L[\![T]\!]$-module

$$\varphi : \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & px \\ & 1 \end{pmatrix} \qquad \varphi : \mathcal{O}_{\mathcal{E}}^+ \longrightarrow \mathcal{O}_{\mathcal{E}}^+$$

$$\sigma^a : \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix} \qquad \sigma^a : \mathcal{O}_{\mathcal{E}}^+ \longrightarrow \mathcal{O}_{\mathcal{E}}^+.$$

Suppose further that we have $\mathcal{O}_L$-linear P^+-action $P^+ \times M \xrightarrow{\text{cts}} M$. Then on M, we define

$$\varphi_M := \begin{pmatrix} p & \\ & 1 \end{pmatrix}(-) : M \longrightarrow M$$

$$\forall a \in \Gamma, \quad \sigma_M^a := \begin{pmatrix} a & \\ & 1 \end{pmatrix}(-) : M \longrightarrow M$$

Lemma: (M, σ_M, φ_M) is a (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}^+$.

Pf: $\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} \quad \forall x \in \mathbb{Z}_p,\ a \in \mathbb{Z}_p\setminus\{0\}.$

§3: Standard presentations

Let $\pi \in \mathrm{Rep}_{tors}(G)$

Let $K := GL_2(\mathbb{Z}_p)$, $Z := Z(G) = \mathbb{Q}_p^\times$, $K_n := 1 + p^n M_2(\mathbb{Z}_p)$, $n \geq 1$

Def: $\mathcal{W}(\pi)$ is the set of $\mathcal{O}_L$-submodules W of π st.

(i) W is KZ-stable

(ii) W is f.g. / $\mathcal{O}_L$

(iii) W generates π over $\mathcal{O}_L[G]$.

Lemma: $\exists n \geq 1$, $\pi^{K_n} \in \mathcal{W}(\pi)$, so in particular $\mathcal{W}(\pi) \neq \emptyset$.

Def: $W \in \mathcal{W}(\pi) \rightsquigarrow$ set

$$I(W) := c\mathrm{Ind}_{KZ}^{G}(W) := \left\{ \phi : G \to W \;\middle|\; \begin{array}{l} \phi(xh) = x\phi(h) \ \forall x \in KZ, h \in G \\ \mathrm{supp}(\phi) \subseteq KZ \backslash G \ \text{finite} \end{array} \right\}$$

$\circlearrowleft$ G via $(g * \phi)(h) := \phi(hg)$

Def: $\forall g \in G$, $v \in W$. $([g,v] : G \to W) \in I(W)$

$$[g,v](h) := \begin{cases} (hg)v & hg \in KZ \\ 0 & hg \notin KZ. \end{cases}$$

If $[g,W] := \{[g,v] \mid v \in W\} \rightsquigarrow I(W) \underset{\mathcal{O}_L}{=} \bigoplus_{g \in G/KZ} [g,W]$

Def:
$$\Phi : I(W) \longrightarrow \pi$$
$$\phi \longmapsto \sum_{g \in G/KZ} g \cdot \phi(g^{-1}) \qquad \Phi([g,v]) = g \cdot v.$$

Lemma: Φ is well-defined, G-equivariant, surjective.

Def: $0 \longrightarrow R(W,\pi) \longrightarrow I(W) \longrightarrow \pi \longrightarrow 0$

Def: $\forall g \in G$, $v \in W \cap g^{-1}W$.

$$r_g(v) := [g,v] - [1,gv] \in R(W,\pi) \qquad \mathcal{W}^0(\pi) \subseteq \mathcal{W}(\pi)$$

We say that W gives a std presentation for π if

$$\langle \{ r_\omega(v) \mid v \in W \cap \omega^{-1}W \} \rangle_{\mathcal{O}_L[G]} = R(W,\pi)$$

$$\omega := \begin{pmatrix} & p \\ 1 & \end{pmatrix}$$

Thm: $\forall \pi \in \mathrm{Rep}_{tors}(G)$, $\mathcal{W}^0(\pi) \neq \emptyset$. (Not true for $GL_2(F)$, $F \supsetneq \mathbb{Q}_p$)

Def: For $W \in \mathcal{W}(\pi)$, define

$$I_{\mathbb{Z}_p}(W) := \sum_{\substack{n\in\mathbb{Z},\, a\in\mathbb{Q}_p \\ a+p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p}} \left[\begin{pmatrix} p^n & a \\ & 1 \end{pmatrix}, W\right] \subset I(W)$$

$(V^\vee := \mathrm{Hom}_{\mathcal{O}_L}(V, L/\mathcal{O}_L))$

$(g*\mu)(-) := \mu(g^{-1}-)$

(Pontryagin dual)

$$D_W^\natural(\pi) := \Phi(I_{\mathbb{Z}_p}(W))^\vee \subseteq \pi^\vee .$$

$\circlearrowleft$ $\mathcal{O}_\mathcal{E}^+$ via $\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}$

Lemma: $W_2 \subseteq W_1$, $W_1, W_2 \in \mathcal{W}(\pi)$

$$\Phi(I_{\mathbb{Z}_p}(W_2)) \xhookrightarrow[\text{index}]{\text{finite}} \Phi(I_{\mathbb{Z}_p}(W_1))$$

In particular, it induces an isomorphism

$$\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_{W_1}^\natural(\pi) \xrightarrow{\sim} \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_{W_2}^\natural(\pi)$$

Def: $D(\pi) := \varprojlim_{W\in\mathcal{W}(\pi)} \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_W^\natural(\pi) \curvearrowleft \mathcal{O}_\mathcal{E}$.

(Montreal functor)

Def: If $W \in \mathcal{W}^0(\pi)$, let

$$D_W^+(\pi) := \left\{ \mu \in \pi^\vee \;\middle|\; \mu|_{\begin{pmatrix} p^n & a \\ & 1\end{pmatrix} \cdot W} \equiv 0 \quad \forall n\in\mathbb{Z},\, a \in \mathbb{Q}_p \text{ s.t. } a+p^n\mathbb{Z}_p \not\subseteq \mathbb{Z}_p \right\} \subseteq \pi^\vee$$

- $D_W^+(\pi) \subset \pi^\vee$ is closed under P^+-action

Prop: If $W \in \mathcal{W}^0(\pi)$, then $D_W^+(\pi) \xhookrightarrow[\text{index}]{\text{finite}} D_W^\natural(\pi)$

$$\underbrace{\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_W^+(\pi)}_{(\varphi,\Gamma)\text{-module}} \xrightarrow{\sim} \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_W^\natural(\pi)$$

so $D(\pi)$ is a (φ,Γ)-module