Definition of the Montreal Functor

- Let $L/\mathbb{Q}_p$ be a fin. ext., with $\mathcal{O}_L \ni \varpi$.
- Let $G := GL_2(\mathbb{Q}_p)$.

Aim: Define a functor

$$\begin{tikzcd} D : \mathrm{Rep}_{tors}(G) \arrow[r] \arrow[dr, dashed, "(\text{Hairnad})"'] & \Phi\Gamma(\mathcal{O}_\mathcal{E}) \\ & \Phi\Gamma^{et}_{tors}(\mathcal{O}_\mathcal{E}) \arrow[u, hook] \end{tikzcd}$$

§1 The Categories

§1.1 $\mathrm{Rep}_{tors}(G)$

Def: $\mathrm{Rep}_{tors}(G)$ is the full subcategory of $\mathcal{O}_L[G]$-mod such that:

(i) Π smooth, and $\forall K \leq_{c.o} G$, Π^K fin. length $/\mathcal{O}_L$
(ii) Π fin. length $/\mathcal{O}_L[G]$
(iii) Π admits a central char.

Rmk: $M \in \mathcal{O}_L$-mod is fin. length
$\iff M$ f.g. + torsion
$\iff M$ finite as a set.

Rmk: (i) can be replaced with
(i)* Π smooth, admissible and $\mathcal{O}_L$-torsion.

§1.2 (φ, Γ)-modules

$$\mathcal{O}_\mathcal{E}^+ := \mathcal{O}_L[\![T]\!] \leq \mathcal{O}_\mathcal{E} = \left\{ \sum_{k \in \mathbb{Z}} a_k T^k \;\middle|\; \begin{array}{l} a_k \in \mathcal{O}_L \\ a_k \to 0 \text{ as } k \to -\infty \end{array} \right\}$$

- $\mathcal{O}_\mathcal{E}$ is a DVR with uniformiser ϖ
- $\mathcal{O}_\mathcal{E}$ given coarsest top s.t.
$\mathcal{O}_\mathcal{E} \twoheadrightarrow k_L((T))$ cts, where
$\uparrow$ topology coming from T-adic on $k_L[\![T]\!]$.
- Fewer open sets than the valuation top.
- Nbd basis: $\varpi^n \mathcal{O}_\mathcal{E} + T^m \mathcal{O}_\mathcal{E}^+$.
- On $\mathcal{O}_\mathcal{E}^+, \mathcal{O}_\mathcal{E}$, have $\mathcal{O}_L$-linear cts

$$\varphi : f(T) \longmapsto f((1+T)^p - 1)$$
$$(a \in \mathbb{Z}_p^\times \overset{\cong}{=} \Gamma) \quad \sigma_a : f(T) \longmapsto f((1+T)^a - 1).$$

Def: Let $R \in \{\mathcal{O}_\mathcal{E}^+, \mathcal{O}_\mathcal{E}\}$. Then a (φ,Γ)-module over R is a triple (M, σ^M, φ^M) where:

(i) M is a top. R-module
(ii) $\varphi^M : M \to M$ is φ-semilinear
(iii) $\forall a \in \Gamma$, $\sigma_a^M : M \longrightarrow M$ σ_a-semilinear action with $\sigma^M : \Gamma \times M \longrightarrow M$ cts
such that, φ^M and σ_a^M all commute.

- Called étale if $\langle \varphi_M(M) \rangle_R = M$
- Called torsion if finite length as R-module.

Write $\Phi\Gamma(R)$ for the category of (φ,Γ)-modules over R, adding subscripts to denote app. full subcategories.

Lemma: $\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} - : \Phi\Gamma(\mathcal{O}_\mathcal{E}^+) \longrightarrow \Phi\Gamma(\mathcal{O}_\mathcal{E})$ is exact, preserves étale, torsion.

- Furthermore, $\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} M = 0$ whenever M is finite length over $\mathcal{O}_L$.

---

§2 Getting a (φ,Γ)-module from a representation

Def: $N_0 := \begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix} \subset P^+ := \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ & 1 \end{pmatrix} \subset G$
(group) (monoid)

- Suppose M is a complete topological $\mathcal{O}_L$-module, with an $\mathcal{O}_L$-linear action of N_0 s.t. $N_0 \times M \longrightarrow M$ is cts.
- Then the $\mathcal{O}_L[N_0]$-module structure extends to an $\mathcal{O}_L[\![N_0]\!]$-module structure, where

$$\mathcal{O}_L[\![N_0]\!] := \varprojlim_{H \leq_{c.o} N_0} \mathcal{O}_L[N_0/H] \hookleftarrow \mathcal{O}_L[N_0].$$

Relation with $\mathcal{O}_\mathcal{E}^+$: There is an $\mathcal{O}_L$-algebra isom:

$$\mathcal{O}_\mathcal{E}^+ = \mathcal{O}_L[\![T]\!] \longleftrightarrow \mathcal{O}_L[\![N_0]\!]$$
$$(1+T) \longleftrightarrow \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

Under this isomorphism

$$\varphi \longleftrightarrow \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & px \\ 0 & 1 \end{pmatrix}\right]$$
$$\sigma_a \longleftrightarrow \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}\right]$$

Now suppose that further, M has an $\mathcal{O}_L$-linear action of P^+, with $P^+ \times M \to M$ cts.

Then we may define:

$$\varphi_M := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} : M \longrightarrow M$$
$$\sigma_M^a := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : M \longrightarrow M.$$

Lemma: (M, φ_M, σ_M) is a (φ,Γ)-module $/\mathcal{O}_\mathcal{E}^+$.

Pf: $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$
$\left(\sigma_M^a(x \cdot m) = \sigma^a(x) \cdot \sigma_M^a(m) \right).$

§3 Standard Presentations

Let $\Pi \in \mathrm{Rep}_{tors}(G)$.

Let $K := GL_2(\mathbb{Z}_p)$, $K_n := 1 + p^n M_2(\mathbb{Z}_p)$, $n \geq 1$.

Def: $\mathcal{W}(\Pi)$ is the set of $\mathcal{O}_L$-s.modules W of Π s.t.

(i) W is KZ-stable
(ii) W is f.gen $/\mathcal{O}_L$ ($\iff$ fin. length)
(iii) W generates Π $/ \mathcal{O}_L[G]$

Lemma: $\exists n \geq 1$ s.t. $\Pi^{K_n} \in \mathcal{W}(\Pi)$.
In particular, $\mathcal{W}(\Pi) \neq \emptyset$.

Def: If $W \in \mathcal{W}(\Pi)$, set

$$I(W) := c\text{-}\mathrm{Ind}_{KZ}^G(W) := \left\{ \phi : G \to W \;\middle|\; \begin{array}{l} \phi(xh) = x\phi(h) \ \forall x \in KZ \\ \mathrm{supp}(\phi) \subset_{KZ} G \text{ finite.} \end{array} \right\}$$

$$(g * \phi)(h) = \phi(hg).$$

- This is a smooth $\mathcal{O}_L[G]$-module.
- We will use the following explicit elements:

Def: Let $g \in G$, $v \in W$. Define $[g,v] \in I(W)$ by

$$[g,v](h) := \begin{cases} (hg) \cdot v & hg \in KZ \\ 0 & \end{cases}$$

$v \neq 0 \to \mathrm{supp}(\phi) = g^{-1} KZ$

Rmk: Setting $[g,W] = \{ [g,v] \mid v \in W \}$,

$$I(W) = \bigoplus_{g \in G/KZ} [g, W]$$ canonically as $\mathcal{O}_L$-modules.

Def: $\Phi : I(W) \longrightarrow \Pi$
$$\phi \longmapsto \sum_{g \in G/KZ} g \cdot \phi(g^{-1})$$

Lemma: Φ is G-equiv, surjective, and independent of the choices of coset representatives.

Example: $\Phi([g,v]) = g \cdot v$.

Def: $0 \to R(W,\Pi) \longrightarrow I(W) \longrightarrow \Pi \longrightarrow 0$

Def: $\forall g \in G$, $v \in W \cap g^{-1}W$,

$$r_g(v) := [g,v] - [1, gv] \in R(W,\Pi)$$

Def: $W \in \mathcal{W}(\Pi)$ is said to give a standard presentation for Π, if

$$R(W,\Pi) = \langle \{ r_\omega(v) \mid v \in W \cap \omega^{-1} W \} \rangle_{\mathcal{O}_L[G]}$$

where $\omega = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

- Defines $\mathcal{W}^0(\Pi) \subset \mathcal{W}(\Pi)$.

The main reason we have to restrict to $GL_2(\mathbb{Q}_p)$ is the follow, which is expected not to hold for $GL_2(F)$, $F > \mathbb{Q}_p$.

Thm: $\mathcal{W}^0(\Pi) \neq \emptyset$. $(\forall \Pi \in \mathrm{Rep}_{tors}(G))$
(Explicitly go through the list of irred. obj of $\mathrm{Rep}_{tors}(G)$)

Now we can come to the definition of $D(\Pi)$.

Def: (for $W \in \mathcal{W}(\Pi)$)
$$I_{\mathbb{Z}_p}(W) := \sum_{\substack{a \in \mathbb{Q}_p, n \in \mathbb{Z} \\ a + p^n \mathbb{Z}_p \subset \mathbb{Z}_p}} \left[\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix}, W \right]$$

$$D_W^\natural(\Pi) := \Phi(I_{\mathbb{Z}_p}(W))^\vee \subset \Pi^\vee$$

$(g * \mu) = \mu(g^{-1} -)$

where $V^\vee := \mathrm{Hom}_{\mathcal{O}_L}(V, L/\mathcal{O}_L)$ is the pontryagin dual.

Lemma: If $W_1, W_2 \in \mathcal{W}(\Pi)$, $+ W_2 \subset W_1$ then

$$\Phi(I_{\mathbb{Z}_p}(W_2)) \hookrightarrow \Phi(I_{\mathbb{Z}_p}(W_1))$$ fin. index.

Cor: If $W_2 \subset W_1$, $W_1, W_2 \in \mathcal{W}(\Pi)$, then:

$$\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_{W_1}^\natural(\Pi) \xrightarrow{\sim} \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_{W_2}^\natural(\Pi)$$

(using that $D_{W_i}^\natural(\Pi)$ is a $\mathcal{O}_\mathcal{E}^+$-module).

- Using that G/KZ is Bruhat Tits tree, can define $W^{[n]} \supset W$, $\forall n \geq 1$, and show $\forall W_1, W_2 \in \mathcal{W}(\Pi)$, $\exists W_1^{[n]} \supset W_2$.

Def: $D(\Pi) := \varprojlim_{W \in \mathcal{W}(\Pi)} \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_W^\natural(\Pi)$. In part., directed set.

- We need to use $\mathcal{W}^0(\Pi)$ to get the (φ,Γ)-module structure.

Def: For $W \in \mathcal{W}^0(\Pi)$, let

$D_W^+(\Pi) \subset \Pi^\vee$ be those
$$= \left\{ \mu \in \Pi^\vee \;\middle|\; \mu|_{\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \cdot W} \equiv 0 \ \forall \begin{array}{l} n \in \mathbb{Z}, a \in \mathbb{Q}_p \\ \text{s.t. } a + p^n \mathbb{Z}_p \not\subset \mathbb{Z}_p \end{array} \right\}$$

Main point:
- $D_W^+(\Pi) \subset \Pi^\vee$ closed under the action of P^+.
- <u>If $W \in \mathcal{W}^0(\Pi)$</u>, then $D_W^+(\Pi) \hookrightarrow D_W^\natural(\Pi)$ is finite index,

so again: $$\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_W^+(\Pi) \xrightarrow{\sim} \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} D_W^\natural(\Pi)$$

Because P^+ action is induced from action on $\Pi^\vee$, all are compatible, and so get well-defined (φ,Γ)-module structure on $D(\Pi)$.