TALK 3

Banach space reps and p-adic LLC

$F/\mathbb{Q}_p$ finite ext.
$\mathcal{O}_F$ ring of int. of F.

8 Feb

Complex LLC

$\{$irred. smooth $\mathbb{C}$-reps of $GL_n(F)\}/\sim \;\leftrightarrow\; \{$n-dim Frob. s.s. WD-reps of $W_F\}/\sim$

$p \neq \ell$-adic LLC

$\{$irred. smooth $\overline{\mathbb{Q}}_\ell$-reps of $GL_n(F)$ which admit a stable lattice$\}/\sim \;\leftrightarrow\; \{$n-dim. Frob s.s. $\overline{\mathbb{Q}}_\ell$-reps of $Gal(\overline{F}/F)\}/\sim$

smooth vectors ↑ $\quad \wr\Vert \quad$ ↓ complete w.r.t. lattice

$\{$top. irred. adm. Banach unitary $\overline{\mathbb{Q}}_\ell$-reps of $GL_n(F)\}/\sim$

Montreal functor = p-adic LLC for $n=2$, $F=\mathbb{Q}_p$.

Can twist complex LLC to make it $Aut(\mathbb{C})$-equivariant.
We get $\overline{\mathbb{Q}}_\ell$-correspondence (fix $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)
Restrict to cts. $\overline{\mathbb{Q}}_\ell$-reps of W_F.
(forget N) $\rightsquigarrow$ cts. W_F-reps $\sigma \leftrightarrow (\rho, N)$

σ extends to $Gal_F \iff LLC(\rho, N)$ has stable $\overline{\mathbb{Q}}_\ell$-lattice.

F-Banach spaces and reps

Def: A norm $\|\cdot\|: V \to \mathbb{R}$ on a F-v.s. V satisfies

- if $0 \neq v \in V$, then $\|v\| > 0$
- $a \in F, v \in V$, $\|av\| = |a|\,\|v\|$
- $v, w, z \in V$, then $\|v-w\| \leq \max\{\|v-z\|, \|z-w\|\}$

Def: A topological F-v.s. is Banach if $\exists\ \|\cdot\|$ on V compatible with top. w.r.t. which V is complete.

Def: G top group. V = F-Banach space, then a rep. of G on V is given by $G \times V \to V$ cts. (stronger than G acts by cts operators) and the G-action is linear. Let $Ban_F(G)$ denote cat. of F-Banach G-reps.

"G profinite $\Rightarrow Ban_F(G)$ abelian" ⟵ not true

Def: V an F-Banach rep of G is unitary if $\exists \|\cdot\|$ w.r.t. which G acts by norm preserving transformations. Write $\mathrm{Ban}_F^u(G)$ for the category of unitary F-Banach reps.

Admissibility

Def: R = ring, G = group, M = R-mod. then a rep. of G on M is a morphism $G \longrightarrow \mathrm{Aut}_R(M)$.

Thm (Maschke): $\mathrm{Rep}_R(G)$ is s.s. $\Leftrightarrow$ R is s.s. + G is finite + $|G| \in R^\times$

G profinite, assume $\exists$ neighbourhood basis of id of open normal neighbourhoods $N \trianglelefteq G$ s.t. $[G:N] \in R^\times$.

R s.s. $\Rightarrow$ smooth R-reps. of G are s.s.

Def. M = smooth R-rep of G (top group) is admissible if $\forall H \leq G$ open, M^H is finitely gen. R-mod.

Thm: G loc. profinite, R s.s., assume $\exists K \leq G$ open compact s.t. $\forall N \leq K$ open $[K:N] \in R^\times$, then $\mathrm{ARep}_R(G)$ is abelian.
($\mathrm{ARep}_R(G)$: category of admissible reps.)

Thm: G loc. pro-p group and p-adic Lie group, then cat. of $\mathcal{O}_F$-torsion reps. of G is abelian.

G loc. pro-p group. V unitary F-Banach rep. of G. Fix $\|\cdot\|$.

$V_0 := \{v \in V \mid \|v\| \leq 1\}$ closed and open
V_0 is G-stable.
V_0 is closed under addition and multiplication by $\mathcal{O}_F$
$\Longrightarrow$ V_0 is $\mathcal{O}_F$-submodule.

So V_0 is an $\mathcal{O}_F$-rep. of G.
Similarly, ϖV_0 is $\mathcal{O}_F$-rep, ϖ uniformizer of F.
$V_0/\varpi V_0$ is a $\mathcal{O}_F/\varpi\mathcal{O}_F$-rep. of G.

Def: V is admissible if $V_0/\varpi V_0$ is admissible. (usual definition)

Categories of the Montréal functor

Norm on a F-v.s. V $\longleftrightarrow$ choice of unit ball V_0 ($\mathcal{O}_F$-sublattice in V)

Def: $L \subseteq V$ is $\mathcal{O}_F$-lattice if it is $\mathcal{O}_F$-submodule containing a basis of V and s.t. it contains no line $\leftarrow$ so cannot be whole space

($\Leftrightarrow$ $\forall v \in V$ $\exists m_1, m_2 \in \mathbb{Z}$, $\varpi^{m_1} v \in L$ and $\varpi^{m_2} v \notin L$)

For example, $\mathcal{O}_F^m \subseteq F^m$ is a lattice.

- Unitary F-Banach rep. is det. by its action on a unit ball.
- F-Banach rep is unitary $\Leftrightarrow$ $\exists$ G-stable unit ball ($\mathcal{O}_F$-lattice)

$\mathrm{Rep}_F^c(G)$ is cat. of unitary F-Banach space reps. with fixed norm + regularity properties.

If $V \in \mathrm{Rep}_F(G)$ equipped with G-stable lattice in $\mathrm{Rep}^c_{\mathcal{O}_F}(G)$, then $V \in \mathrm{Rep}_F^c(G)$.

Take the unit ball $V_0 \in \mathrm{Rep}_{\mathcal{O}_F}(G)$.

(i) V_0 is $\mathcal{O}_F$-torsion free

(ii) V_0 is Hausdorff and complete w.r.t. p-adic topology on V_0 (top. given by $V_0 \supseteq pV_0 \supseteq p^2 V_0 \supseteq \dots$)

(iii) regularity properties

Category of all such $\mathcal{O}_F$-reps of G is $\mathrm{Rep}^c_{\mathcal{O}_F}(G)$.

<u>Regularity</u>: $\forall m$ $V_0/p^m V_0 \in \mathrm{Rep}_{tors}(G) \subseteq \mathrm{Rep}_{\mathcal{O}_F}(G)$

$M \in \mathrm{Rep}_{tors}(G)$ if

(i) M (smooth and) admissible

(ii) M is $\mathcal{O}_F$-torsion

(iii) M has finite length

(iv) M admits a central character (i.e. $\exists$ morphism $\psi: Z(G) \to \mathcal{O}_F^\times$ st. $\forall z \in Z(G)$ $\forall m \in M$, $z \cdot m = \psi(z) m$)