TALK 2: Representations of $\mathrm{Gal}(\bar{F}/F)$ and mod p LLC for $GL_2(\mathbb{Q}_p)$ 31 Jan

1. Galois groups

Let $F/\mathbb{Q}_p$ finite ext., with uniformizer ϖ, residue field $k_F \cong \mathbb{F}_q$.
Let $\bar{F}$ = alg. closure of F. $\Gamma_F := \mathrm{Gal}(\bar{F}/F)$.

$$\begin{tikzcd}
\bar{F} \arrow[d, dash] \arrow[d, bend right, dash, "P_F"'] \arrow[dd, bend left=60, dash, "I_F"] \arrow[ddd, bend right=80, dash, "\Gamma_F"'] \\
F^{tr} := \bigcup_{(n,p)=1} E_n \arrow[d, dash] \arrow[d, bend right, dash, "I_F/P_F"'] \\
F^{ur} := \bigcup_{n\geq 1} F_n \arrow[d, dash] \arrow[d, bend left, dash, "\Gamma_F/I_F = \hat{\mathbb{Z}}"] \\
F
\end{tikzcd}$$

$E_n = F^{ur}(\varpi^{1/n})$ unique ext. of deg n over F^{ur} if $(n,p)=1$.

$F_n = F(\zeta_{q^n-1})$ unique ext. of deg n over F

$I_F = \mathrm{Gal}(\bar{F}/F^{ur})$ = inertia subgroup.
$P_F = \mathrm{Gal}(\bar{F}/F^{tr})$ = wild inertia (pro-p + quite big)

By Kummer theory, $\exists$ canonical isomorphism (independent of choice of ϖ and $\varpi^{1/n}$)

$$\mathrm{Gal}(E_n/F^{ur}) \xrightarrow{\sim} \mu_n(\bar{F})$$
$$\sigma \longmapsto \sigma(\varpi^{1/n})/\varpi^{1/n}$$

$$\mathrm{Gal}(F^{tr}/F^{ur}) = \varprojlim_{(n,p)=1} \mathrm{Gal}(E_n/F^{ur}) = \varprojlim_{(n,p)=1} \mu_n(\bar{F}) \cong \prod_{\ell\neq p} \mathbb{Z}_\ell$$

fix compatible system $(\zeta_n)_{(n,p)=1}$

$$\mathrm{Gal}(F^{tr}/F^{ur}) = \varprojlim_n \mathrm{Gal}(E_{q^n-1}/F^{ur}) = \varprojlim_n \mu_{q^n-1}(\bar{F}) = \varprojlim_n [k_{F_n}^\times] = \varprojlim_n k_{F_n}^\times$$
$$\cong \varprojlim_n \mathbb{F}_{q^n}^\times$$

2. Galois reps over $\bar{\mathbb{F}}_p$

2.1 1-dimensional $\mathrm{Gal}(\bar{F}/F)$-reps

Lemma 1: Any cts character $\theta: I_F \to \bar{\mathbb{F}}_p^\times$ factors as

$$\theta: I_F \twoheadrightarrow I_F/P_F = \varprojlim_n k_{F_n}^\times \cong \varprojlim_n \mathbb{F}_{q^n}^\times \longrightarrow \mathbb{F}_{q^m}^\times \xrightarrow{\bar{\theta}} \bar{\mathbb{F}}_p^\times$$

Pf: • First, want to show $\theta(P_F)$ is finite.
• $\ker\theta$ open by cts. So $(\ker\theta)\cap P_F$ open and normal in P_F.
So $P_F/(\ker\theta)\cap P_F$ is a p-group (finite)
• So $\theta(P_F)$ finite $\Rightarrow$ $\theta(P_F)=1$ $\Rightarrow$ $P_F \leq \ker\theta$.
• $\theta: I_F/P_F \longrightarrow \bar{\mathbb{F}}_p^\times$ has open kernel still,
so θ factors through finite quotient of I_F/P_F. ✓

Definition 2 (Serre's fundamental characters): For $n \geq 1$,

$$\omega_n : I_F \twoheadrightarrow I_F / P_F = \varprojlim_m k_{F_m}^\times \cong \varprojlim_m \mathbb{F}_{q^m}^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times \hookrightarrow \overline{\mathbb{F}}_p^\times .$$

Proposition 3:

(a) If $m \mid n$, then $\omega_n^{1+q^m+q^{2m}+\cdots+q^{(\frac{n}{m}-1)m}} = \omega_m$

(b) $\omega_n^{q^n-1} = \mathbb{1}$, the trivial character.

(c) Every cts character $\theta : I_F \to \overline{\mathbb{F}}_p^\times$ can be written as ω_n^r for some $n \geq 1$ and $0 \leq r < q^n - 1$ primitive

r is not divisible by $\frac{q^n-1}{q^d-1} = 1 + q^d + \cdots + q^{(\frac{n}{d}-1)d}$ for some proper divisor $d \mid n$.

Pf: (a) See notes.

(b) $\omega_n(I_F) \subseteq \mathbb{F}_{q^n}^\times$

(c) By Lemma 1,

$$\theta : I_F \longrightarrow \cdots \longrightarrow \mathbb{F}_{q^n}^\times \xrightarrow{\bar\theta} \overline{\mathbb{F}}_p^\times$$

But $\bar\theta(\mathbb{F}_{q^n}^\times) \subseteq \mathbb{F}_{q^n}^\times$, so $\bar\theta \in \mathrm{Hom}(\mathbb{F}_{q^n}^\times, \mathbb{F}_{q^n}^\times) = \mathrm{Hom}(C_{q^n-1}, C_{q^n-1})$
so $\bar\theta$ is a power of the identity map $\underbrace{\mathbb{F}_{q^n}^\times \longrightarrow \mathbb{F}_{q^n}^\times}_{= \omega_n}$.

Lemma 4: Let φ be a lift of Frob to Γ_F / P_F. Let $\tau \in I_F / P_F \trianglelefteq \Gamma_F / P_F$.
Then $\varphi \tau \varphi^{-1} = \tau^q$ in Γ_F / P_F

Lemma 5: $\omega_n : I_F \longrightarrow \overline{\mathbb{F}}_p^\times$ can be extended (non-uniquely) to Γ_F
$\iff n = 1$.

Pf: ($\Rightarrow$) Suppose ω_n extends to Γ_F. Let $\varphi \in \Gamma_F$ lift Frob, let $\tau \in I_F$.

$$\begin{aligned} \omega_n(\tau) &= \omega_n(\varphi)\,\omega_n(\tau)\,\omega_n(\varphi^{-1}) \\ &= \omega_n(\varphi\tau\varphi^{-1}) \\ &= \omega_n(\tau)^q \qquad (\omega_n \text{ factors through } I_F/P_F) \end{aligned}$$

So ω_n has image in $\mathbb{F}_q^\times \Rightarrow \mathbb{F}_{q^n}^\times \subseteq \mathbb{F}_q^\times \Rightarrow n = 1$.

Corollary 6: Any cts character $\chi : \Gamma_F \longrightarrow \overline{\mathbb{F}}_p^\times$ is of the form

$$\omega_1^r \cdot \mu_\lambda \qquad (0 \leq r < q - 1)$$

where $\mu_\lambda : \Gamma_F \twoheadrightarrow \Gamma_F / I_F \longrightarrow \overline{\mathbb{F}}_p^\times$ sends $\varphi \mapsto \lambda$.

2.2 n-dimensional $\mathrm{Gal}(\overline{F}/F)$-reps

Proposition 7: Let $(\rho, V) : \Gamma_F \to GL_n(\overline{\mathbb{F}}_p)$ cts irrep. Then

$$\rho|_{I_F} = \bigoplus_{i=1}^{n} \omega_{m_i}^{r_i} \qquad (0 \leq r_i < q^{m_i} - 1)$$

Pf: Since P_F is pro-p, ρ is smooth mod p rep $\Longrightarrow$ $V^{P_F} \neq 0$. some lemma

$P_F \triangleleft \Gamma_F \Rightarrow V^{P_F} \circlearrowleft \Gamma_F$; V irred $\Rightarrow V = V^{P_F}$

So $\rho : \Gamma_F \twoheadrightarrow \Gamma_F / P_F \longrightarrow GL_n(\overline{\mathbb{F}}_p)$.

$$\rho|_{I_F} : I_F \twoheadrightarrow I_F/P_F \longrightarrow GL_n(\overline{\mathbb{F}}_p).$$

$\rho|_{I_F}$ cts $\Rightarrow$ $\rho|_{I_F}$ factors through finite quotient of $I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$

$\Rightarrow$ $\rho|_{I_F}$ factors through H, $(\# H, p) = 1$.

Maschke + Schur $\Rightarrow$ result.

Proposition 8: Let $\Gamma_{F_n} = \langle I_F, \varphi^n \rangle$.

(i) φ^{-1} acts by n-cycle on $\{\omega_{m_i}^{r_i}\}$, so $\varphi^n(\omega_{m_i}^{r_i}) \subseteq \omega_{m_i}^{r_i}$ $\forall i = 1, \dots, n$.
Deduce $m_i \mid n$ for all i (so can choose $m_i = n$ $\forall i$)

(ii) If $m \mid n$, then ω_m extends to Γ_{F_n} by setting $\omega_n(\varphi^n) = 1$. generalizes Lemma 5

(iii) $\exists \kappa_\lambda : \Gamma_{F_n} \twoheadrightarrow \Gamma_{F_n}/I_F \longrightarrow \overline{\mathbb{F}}_p^\times$, $\kappa_\lambda(\varphi^n) = \lambda$.

$$\rho|_{\Gamma_{F_n}} = \bigoplus_{i=1}^{n} \left(\omega_n^{rq^{i-1}} \kappa_\lambda\right)$$

Pf: (i) • Action by n-cycle is by irreducibility of ρ.
• Let $v \in \omega_{m_i}^{r_i}$ and $\tau \in I_F$. Then

$$\tau(\varphi^{-1} v) = \varphi^{-1}\tau^q(v) = \omega_{m_i}^{r_i}(\tau)^q(\varphi^{-1} v)$$

So $\varphi^{-1} v \in \omega_{m_i}^{q r_i}$. Then

$$\varphi^{-n} v \in \omega_{m_i}^{r_i} \Rightarrow \omega_{m_i}^{r_i} = \omega_{m_i}^{r_i q^n}$$
$$\Rightarrow \omega_{m_i}^{r_i}(I_F) \in \mathbb{F}_{q^n}^\times$$
$$\Rightarrow \text{can set } m_i = n \ \forall i$$

(ii) Just try to do it.

(iii) Let $m_1 = n$ and $r = r_1$. By part (i),

$$\rho|_{\Gamma_{F_n}} = \omega_n^r \kappa_\lambda \oplus \varphi^{-1}(\omega_n^r \kappa_\lambda) \oplus \cdots \oplus \varphi^{-(n-1)}(\omega_n^r \kappa_\lambda)$$
$$= \omega_n^r \kappa_\lambda \oplus \omega_n^{rq} \kappa_\lambda \oplus \cdots \oplus \omega_n^{rq^{n-1}} \kappa_\lambda.$$

Corollary 9: Let $\rho : \Gamma_F \longrightarrow GL_n(\overline{\mathbb{F}}_p)$ cts irrep. Then

$$\rho \cong \mathrm{Ind}_{\Gamma_{F_n}}^{\Gamma_F}(\omega_n^r \kappa_\lambda) \quad \begin{pmatrix} \text{some } 0 \leq r < q^n - 1 \\ \text{some } \lambda \in \overline{\mathbb{F}}_p^\times \end{pmatrix}$$

Corollary 10: Let $\rho : \Gamma_{\mathbb{Q}_p} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$ cts irrep. Then

$$\rho \cong \mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^r \kappa_\lambda) \quad \begin{pmatrix} 0 \leq r < p^2 - 1 \\ \lambda \in \overline{\mathbb{F}}_p^\times \end{pmatrix}$$

Theorem 11 (Thm 1.1 of Berger): Let $\rho : \Gamma_{\mathbb{Q}_p} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$ cts irrep. Then

$$\rho \cong \rho(r,\chi) := \mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^{r+1}) \otimes \chi$$

- $r \in \{0, \ldots, p-1\}$
- $\chi : \Gamma_{\mathbb{Q}_p} \longrightarrow \overline{\mathbb{F}}_p^\times$ smooth.

$$\mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^r \kappa_\lambda) \cong \mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^r) \otimes \mu_{\lambda_0} \quad ; \quad \mu_{\lambda_0}|_{\Gamma_{\mathbb{Q}_{p^2}}} = \kappa_\lambda$$

discard this at expense of twist

- $\mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^r)$ reducible $\Leftrightarrow (p+1) \mid r$
- $\mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^r) \cong \mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^{r-(p+1)}\omega) \cong \mathrm{Ind}_{\Gamma_{\mathbb{Q}_{p^2}}}^{\Gamma_{\mathbb{Q}_p}}(\omega_2^{r-(p+1)}) \otimes \omega$

$$\left(\omega := \omega_1 = \omega_2^{1+p}\right)$$

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3. (semi-simple) mod p LLC for $GL_2(\mathbb{Q}_p)$

Let $G = GL_2(\mathbb{Q}_p)$, $K = GL_2(\mathbb{Z}_p)$. $Z \cong \mathbb{Q}_p^\times$ centre of G.

$$\pi(r,\lambda,\chi) := \frac{c\text{-}\mathrm{Ind}_{KZ}^G(\mathrm{Sym}^r \overline{\mathbb{F}}_p^2)}{(T_p - \lambda)} \otimes (\chi \circ \det) \qquad \left(\begin{array}{l} 0 \le r < p-1 \\ \chi : \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{F}}_p^\times \\ \lambda \in \overline{\mathbb{F}}_p \end{array}\right)$$

Let $\chi, \omega : \begin{cases} \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times \\ \Gamma_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times \end{cases}$ via LCFT. and $\lambda \in \overline{\mathbb{F}}_p$

- For $r \in \{0, \ldots, p-1\}$, $\quad \rho(r,\chi) \longleftrightarrow \pi(r,0,\chi)$
- For $r \in \{0, \ldots, p-2\}$, $\lambda \neq 0$,

$$(\omega^{r+1}\mu_\lambda \oplus \mu_{1/\lambda}) \otimes \chi \longleftrightarrow \pi(r,\lambda,\chi)^{ss} \oplus \pi(p-3-r, 1/\lambda, \omega^{r+1}\chi)^{ss}$$

Note: Objects on Galois side have determinant $\omega^{r+1}\chi^2$.
Objects on automorphic side have central character $\omega^r \chi^2$.