TALK 1. Mod p representations of $GL_2(\mathbb{Q}_p)$. 1/25/24.

Goal. Classify all smooth, admissible, irreducible representations of $GL_2(\mathbb{Q}_p)$ over k = finite ext. of $\mathbb{F}_p$, $p \neq 2$.

Def. • A rep. of G over k : $\pi : G \longrightarrow GL(V)$, $V = k$-v.sp.
- Smooth if $\mathrm{Stab}_G(v) \subseteq G$ is open $\forall\, v \in V$
- Admissible if $\dim_k (V^I) < \infty$ $\forall$ compact open $I \leq G$

Note: • over $\mathbb{C}$, smooth, irred $\Longrightarrow$ admissible,
• over k, $\not\Longrightarrow$.

Let: $K_1 := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p \right\} \subseteq I_1 = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod p \right\} \subseteq K := GL_2(\mathbb{Z}_p)$

Prop. (V, π) smooth, non-zero, over k
- $V^{K_1} \neq 0$ & $V^{I_1} \neq 0$ [depth 0]
- V finite-dimensional $\Rightarrow$
  $V = \chi \circ \det$, $\chi : \mathbb{Q}_p^\times \longrightarrow k_E^\times$ s.t. $\chi(1 + p\mathbb{Z}) = 1$ [depth 0]
- V admissible $\Longleftrightarrow$ $\dim_c V^{I_1} < \infty$.

Prop.
$$\left\{ \begin{matrix} \text{smooth irred.} \\ \text{reps of } K \end{matrix} \right\}_{/\cong} \longleftrightarrow \left\{ \begin{matrix} \text{irred. reps} \\ \text{of } GL_2(\mathbb{F}_p) \end{matrix} \right\}_{/\cong}$$
$$W \longmapsto W = W^{K_1} \circlearrowleft K/K_1 \cong GL_2(\mathbb{F}_p)$$
$$W \longleftarrow\!\!\shortmid\; W$$

Rmk. Only reps of $GL_2(\mathbb{F}_p)$ over k are $W_{r,s} := \mathrm{Sym}^r(k^2) \otimes \det^s$
(No "cuspidal" reps like over $\mathbb{C}$ ...) $0 \leq r \leq p-1$, $0 \leq s \leq p-1$.

Key definition. For W = smooth irrep. of K (extend to trivial Z-action)

- $c\text{-}\mathrm{Ind}_{KZ}^{G}(W) := \left\{ f : G \longrightarrow W \;\middle|\; \begin{array}{l} \bullet\ f(kzg) = k\, f(g) \\ \bullet\ \text{locally constant} \\ \bullet\ \text{image of } \operatorname{supp} f \text{ in} \\ \quad KZ \backslash G \text{ is compact} \end{array} \right\}$

  compact induction

- $\mathcal{H}_{KZ}(W) := \mathrm{End}_G\left(c\text{-}\mathrm{Ind}_{KZ}^{G}(W)\right)$ Hecke algebra

$$\cong \left\{ \varphi : G \longrightarrow \mathrm{End}(W) : \begin{array}{l} \bullet\ \varphi(kzgk'z') = k \circ \varphi(g) \circ k' \\ \bullet\ KZ \backslash \operatorname{supp}(\varphi) / KZ \text{ compact} \end{array} \right\}$$

with convolution.

Thm (Herzig, mod p Satake). $\mathcal{H}_{KZ}(W) \cong k[T_p]$,
(Barthel–Livné)
T_p corresp. to $\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$

Note: giving $\lambda : \mathcal{H}_{KZ}(W) \longrightarrow k \iff$ giving $a_p \in k$, T_p - eigenvalue.

Def. Given $0 \le r \le p-1$, $a_p \in k$, $\chi : \mathbb{Q}_p^\times \longrightarrow k^\times$ smooth character,

$$\pi(r, \lambda, \chi) := \frac{c\text{-}\mathrm{Ind}_{KZ}^{G}(\mathrm{Sym}^r k)}{(T_p - a_p)} \otimes (\chi \circ \det).$$

Note: given $a_p \in k^\times$, $\mu_{a_p} : \mathbb{Q}_p^\times \longrightarrow k^\times$

$$p \longmapsto a_p$$
$$\mathbb{Z}_p^\times \longmapsto 1$$

Thm (Barthel–Livné, Breuil). (ω = cyclotomic character)

1. $\pi(r, a_p, \chi)$ is smooth and admissible, central char. $\omega^r \chi^2$
2. $\pi(r, a_p, \chi)$ is irreducible unless $a_p = \pm 1$ and $r = 0, p-1$
3. For $a_p = \pm 1$, $r = 0, p-1$, we have:

$$0 \longrightarrow St \otimes (\chi \mu_{a_p} \circ \det) \longrightarrow \pi(0, a_p, \chi) \longrightarrow \chi \mu_{a_p} \circ \det \longrightarrow 0$$

$$0 \longrightarrow \chi \mu_{a_p} \circ \det \longrightarrow \pi(p-1, a_p, \chi) \longrightarrow St \otimes (\chi \mu_{a_p} \circ \det) \longrightarrow 0$$

where $St := \dfrac{\{\text{loc. const. } \mathbb{P}^1(\mathbb{Q}_p) \longrightarrow k\}}{\{\text{const. } \mathbb{P}^1(\mathbb{Q}_p) \longrightarrow k\}}$ (Steinberg rep.)

is irreducible.

4. $\pi(r, a_p, \chi)$ is "supercuspidal" $\Leftrightarrow a_p = 0$. (supersingular)
5. These are all smooth irred. adm. reps of $GL_2(\mathbb{Q}_p)$ over k.

Def. (π, V) = irred. adm. rep. of G

- **supercuspidal** if not subquotient of any $\mathrm{Ind}_B^G(\psi_1 \otimes \psi_2)$
  $\psi_1, \psi_2 : \mathbb{Q}_p^\times \longrightarrow k$, smooth.
- Jacquet module V_N $\quad [V_N = V / \langle n.v - v \mid v \in V, n \in N \rangle]$

Note: $V_N \neq 0 \Leftrightarrow V \cong$ subrep. of $\mathrm{Ind}_B^G(\psi_1 \otimes \psi_2)$, some ψ_1, ψ_2.

Thm.
1. $\mathrm{Ind}_B^G(\psi_1 \otimes \psi_2)$ is smooth & admissible, central char. $\psi_1 \cdot \psi_2$.
2. $\mathrm{Ind}_B^G(\psi_1 \otimes \psi_2)$ irreducible if $\psi_1 \neq \psi_2$, no intertwiners between them
3. If $\psi_1 = \psi_2 = \psi$, then $\exists$ non-split ses:

$$0 \longrightarrow \psi \circ \det \longrightarrow \mathrm{Ind}_B^G(\psi \otimes \psi) \longrightarrow St \otimes \psi \circ \det \longrightarrow 0$$

4. St is <u>not</u> supercuspidal but $V_N = 0$.

Comparison to reps over $\mathbb{C}$:

- in 2. also reducible if $\psi_1 = \psi_2 \cdot |\cdot|^{\pm 2}$
- in 3. also have:

$$0 \longrightarrow \mathrm{St} \otimes \psi \circ \det \longrightarrow \mathrm{Ind}_B^G(\psi \delta_B \otimes \psi \delta_B^{-1}) \to \psi \circ \det \longrightarrow 0.$$

$\Rightarrow$ supercuspidal $\Leftrightarrow V_N = 0$

- to get all supercuspidals, need "lower depth", i.e. $\overbrace{c\text{-}\mathrm{Ind}_{K_r}^G(W)}^{\text{already irreducible!}}$

$K_r = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p^r \right\}$, $W =$ rep. of K_r.

Comparison between two constructions of non-supercuspidals.

Cor. For $a_p \neq 0$, $r \in \{0, \dots, p-1\}$, $\chi : \mathbb{Q}_p^\times \longrightarrow k$ smooth

$$\pi(r, a_p, \chi)^{ss} \cong \mathrm{Ind}_B^G(\chi \mu_{1/a_p}, \chi \omega^r \mu_{a_p})^{ss}.$$

Final result: Which supercuspidals are isomorphic?

Thm. $\pi(r, 0, \chi) \cong \pi(r, 0, \chi\mu_{-1}) \cong \pi(p-1-r, 0, \chi\omega^r) \cong \pi(p-1-r, 0, \chi\omega^r\mu_{-1})$

Why not $GL_n(F)$ for $F/\mathbb{Q}_p$ finite?

- Herzig: → classification of non-supercuspidal reps in terms of parabolic induction ✓
  → mod p Satake ✓
- In fact, generalizes to any conn. red. G
  (Abe, Abe–Henniart–Herzig–Vignéras)

[Better than $\mathbb{C}$-coefficients!]

- Classification of supercuspidals? Open even for $GL_2(F)$, $F \neq \mathbb{Q}_p$:

$$\frac{c\text{-}\mathrm{Ind}_{KZ}^{G}(W)}{(T_p)} \text{ is (in general)}$$

- not admissible
- ∞-length
- has ∞ many irred. adm. s.s. quotients
- has irred. non-admissible quotients

Why? Proof of irred. for $\frac{c\text{-}\mathrm{Ind}_{KZ}^{G}(W)}{(T_p)}$ for $GL_2(\mathbb{Q}_p)$ involves taking invariants by $U_0 = \begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix} \cong \mathbb{Z}_p$, which is not exact, but $\mathbb{Z}_p$ has coh. dim. $1 \Rightarrow$ never have to go above H^1.

For $F \neq \mathbb{Q}$, coh. dim grows with $[F : \mathbb{Q}_p]$

$\Rightarrow$ hard to control the rep. theory.

[Worse than $\mathbb{C}$-coefficients.]

- Fascinating work of Paskunas, Paskunas–Breuil, Hu ...

  for one Galois rep ρ, $\exists$ ∞-many reps π of $GL_2(F)$ "corresponding to it".