The Ihara-Langlands-Kottwitz method for modular curves

29 Feb

Modular forms wt. $k \geq 2$ $\longrightarrow$ étale coh. of modular curves

Hecke e.vals $\longleftrightarrow$ Trace of Frob. elements.

$\uparrow$ Eichler-Shimura relation (geometric)

To do more general constructions, coh. of Shimura varieties (say for unitary groups)

Still want to get

$$\rho_{\pi,\ell} = H^d_{et}(Y_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)[\text{Hecke eigenspace}]$$

Have Eichler-Shimura relation (Wedhorn?), doesn't give enough information to compute char. poly. of Frobenius. (for unramified primes p)

char poly. of $\rho_{\pi,\ell}(\mathrm{Frob}_p) \longleftrightarrow$ Satake parameter of π_p.

In general, have to use a different method.

Ihara: computed Hasse-Weil zeta function of modular curves in terms of L-functions of modular forms.

Refined by Langlands to give a new proof of relation

$$\mathrm{tr}\, \rho_f(\mathrm{Frob}_p) = a_p(f).$$

(not using Eichler-Shimura)

Kottwitz carried out a big generalization of Langlands' methods.

Today: modular curves case, following

Scholze: "L-K method for modular curves" (good reduction case)

$Y_m / \mathbb{Z}[\frac{1}{m}]$, $m \geq 3$.

Level: (adelic) $K_m \leq GL_2(\hat{\mathbb{Z}})$.

$$K_m = \{\gamma \mid \gamma \equiv \mathrm{Id} \bmod m\}$$

$$Y_m(\mathbb{C}) = GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) / \mathbb{R}_{>0}SO_2(\mathbb{R})K_m$$

$$= \coprod_{(\mathbb{Z}/m\mathbb{Z})^\times} \Gamma(m)\backslash\Gamma .$$

$$Y_m(R) = \left\{ \text{ell. cur. } E/R \;,\; R = \mathbb{Z}[\tfrac{1}{m}]\text{-algebra} \right\}$$

$$\eta_m : (\mathbb{Z}/m\mathbb{Z})^{\oplus 2}_{\mathrm{Spec}\,R} \xrightarrow{\sim} E[m]$$

$$\eta_m \xrightarrow{\text{Weil pairing}} \text{primitive } m^{\text{th}} \text{ root of unity} .$$

$Y_{m,\mathbb{Q}}$ connected, not geometrically connected.

We'll compute:

$$\mathrm{Tr}(\mathrm{Frob}_p^r, H^i_{et,c}(Y_{m,\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)) \quad \text{for } p \nmid m\ell .$$

also need to compose with Hecke operator away from p.

Thm 1 (Grothendieck-Lefschetz trace formula)

$$\sum_{i=0}^{2} (-1)^i \mathrm{Tr}(\mathrm{Frob}_p^r, H^i_c(Y_{m,\mathbb{F}_p}, \mathbb{Q}_\ell)) = \underbrace{\# Y_m(\mathbb{F}_{p^r})}$$

we'll count this (Langlands-Rapoport conjecture)

compare this with output of Artin-Selberg trace formula

$\updownarrow$

traces of Hecke operators.

actually: $H^1_c \otimes \bar{\mathbb{Q}}_\ell$

What we expect ($r=1$):

$$H^0_c = 0 \,,\; \boxed{H^1_c} = \bigoplus_{\substack{f \text{ cusp} \\ \text{Hecke e.forms} \\ \text{of level } K_m \\ \text{wt. } 2}} e_{f,\ell} \oplus \left(\begin{matrix}\text{contribution} \\ \text{from cusps}\end{matrix}\right) ,\; H^2_c = \mathbb{Q}_\ell[(\mathbb{Z}/m\mathbb{Z})^\times](-1)$$

$\circlearrowleft$ $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\mathrm{Tr}(\mathrm{Frob}_p|_{e_{f,\ell}}) = a_p(f)$$

$$\frac{1}{2}\,\overset{''}{\mathrm{Tr}}(T_p|_{e_{f,\ell}})$$

$\underline{r=1}$: $\mathrm{Tr}(\mathrm{Frob}_p) = \frac{1}{2}\mathrm{Tr}(T_p)$ + explicit "error term".

Fix $\iota : \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$.

$$x \in Y_m(\mathbb{F}_{p^r}) \longleftrightarrow \left\{ (E_x/\mathbb{F}_{p^r}, \eta_m) \;/\; \text{iso.} \right\}$$

Fix $E_0/\mathbb{F}_{p^r}$. Count x with $E_x \overset{\text{isog.}}{\sim} E_0/\mathbb{F}_{p^r}$.

⟶ Then we'll parameterize the isogeny classes.

How do we describe isogenies from E_0?

$$\begin{tikzcd} K \subseteq E_0 \arrow[r, two heads] & E. \\ \text{FFGS}/\mathbb{F}_{p^r} \arrow[u] & \end{tikzcd}$$

$$K = \prod_{\substack{v \text{ finite} \\ \text{places of } \mathbb{Q}}} K[v^\infty] \circlearrowright \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^r})$$

$$K[v^\infty](\overline{\mathbb{F}}_p) \subseteq T_v E_0 \longrightarrow T_v E, \quad v \nmid p.$$

Fix $E_0 \xrightarrow{f} E$ isogeny.

$$L := f^*\left(H^1_{\acute{e}t}(E_{\overline{\mathbb{F}}_p}, \hat{\mathbb{Z}}^p) = \prod_{v \neq p} \underbrace{H^1_{\acute{e}t}(E_{\overline{\mathbb{F}}_p}, \mathbb{Z}_v)}_{(T_vE)^*} \right)$$

L is a lattice in $H^1_{\acute{e}t}(E_{0,\overline{\mathbb{F}}_p}, \hat{\mathbb{Z}}^p) \otimes \mathbb{Q} =: H^p$ ⟵ $\mathbb{A}_f^p$-module

$\updownarrow$ Collection of lattices L_v in $H^1_{\acute{e}t}(E_{0,\overline{\mathbb{F}}_p}, \mathbb{Q}_v)$

$\cup\!|$

L_v vs. $H^1_{\acute{e}t}(E_{0,\overline{\mathbb{F}}_p}, \mathbb{Z}_v)$

captures the v-adic part of f.

L is stable under Frob_p^r (since f defined over $\mathbb{F}_{p^r}$)

$$H_p := \underbrace{H^1_{cris}(E_0/W(\mathbb{F}_{p^r}))}_{} \left[\tfrac{1}{p}\right] = f^*\left(H^1_{cris}(E/W(\mathbb{F}_{p^r}))\right), \quad F, V \text{ invariant}$$

rank 2 $W(\mathbb{F}_{p^r})$-module
with σ-semilinear operator F,
↰ p-power in $\mathbb{F}_{p^r}$
σ^{-1}-s.lin. op. V with $VF = FV = p$

ptm: $(E_x, \eta_m) \longleftrightarrow x \in Y_m(\mathbb{F}_{p^r})$

$f: E_0 \longrightarrow E_x$

$$\rightsquigarrow L \subseteq H^p, \; \Lambda \subset H_p, \; \phi: (\mathbb{Z}/m\mathbb{Z})^2 \xrightarrow{\sim} L/mL$$

Gives us a map as follows:

$$Y^p := \left\{ L \subset H^p \;\middle|\; \begin{array}{l} \hat{\mathbb{Z}}^p\text{-lattice, } \mathrm{Frob}_p^r\text{-stable} \\ \phi : (\mathbb{Z}/m\mathbb{Z})^2 \xrightarrow{\sim} L/mL \end{array} \right\}$$

also Frob_p^r -invariant

$$Y_p := \left\{ \Lambda \subset H_p \;\middle|\; \begin{array}{l} W(\mathbb{F}_{p^r})\text{-lattices,} \\ F, V\text{-stable} \end{array} \right\}$$

$$\left\{ x \in Y_m(\mathbb{F}_{p^r}) \;\middle|\; E_x \text{ isog. with } E_0 \right\} \xrightarrow{\sim} \Gamma \backslash (Y^p \times Y_p)$$

$$\Gamma = \left(\mathrm{End}(E_0) \otimes \mathbb{Q}\right)^\times$$

Thm 2: This is a bijection.

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§3: $\#(\Gamma \backslash Y^p \times Y_p)$ is computed orbital integrals

$$\begin{tikzcd} \mathrm{Frob}_p^r \arrow[d, mapsto] \arrow[r, phantom, "\circlearrowright"] & H^p := H^1_{et}(E_{0,\bar{\mathbb{F}}_p}, \mathbb{A}_f^p) \\ \gamma & (\mathbb{A}_f^p)^{\oplus 2} \arrow[u, phantom, "\cong" rotate=90] \\ GL_2(\mathbb{A}_f^p) \arrow[u, phantom, "\in" rotate=90] & \end{tikzcd}$$

Interested in Y^p, $L \subset (\mathbb{A}_f^p)^{\oplus 2}$, ϕ
γ preserves the lattice ϕ.

$$G_\gamma(\mathbb{A}_f^p) = \left\{ g \in GL_2(\mathbb{A}_f^p) \;\middle|\; g^{-1}\gamma g = \gamma \right\}$$

$$H_p \cong \mathbb{Q}_{p^r}^{\oplus 2}. \qquad F \longmapsto \delta \in GL_2(\mathbb{Q}_{p^r})$$

$$F(a_1 e_1 + a_2 e_2) = \delta \begin{pmatrix} \sigma a_1 \\ \sigma a_2 \end{pmatrix}, \qquad F = \delta\sigma.$$

$$G_\delta(\mathbb{Q}_p) = \left\{ h \in GL_2(\mathbb{Q}_{p^r}) \;\middle|\; h^{-1}\delta^\sigma h = \delta \right\} \quad \text{(twisted centralizer)}$$

$$f^p = \mathbb{1}_{K_m^p} \in C_c^\infty(GL_2(\mathbb{A}_f^p))$$

$$\phi_{p,0} = \mathbb{1}_{GL_2(W(\mathbb{F}_{p^r})) \begin{pmatrix} p & \\ & 1 \end{pmatrix} GL_2(W(\mathbb{F}_{p^r}))} \in C_c^\infty(GL_2(\mathbb{Q}_{p^r}))$$

$$O_\gamma(f^p) := \int_{G_\gamma(\mathbb{A}_f^p) \backslash GL_2(\mathbb{A}_f^p)} f(g^{-1}\gamma g)\, dg$$

$TO_{\delta\sigma}(\phi_{p,0})$:= similar.

Thm: $\#(\Gamma \backslash Y^p \times Y_p) = (\mathrm{vol}) \cdot O_\gamma(f^p)\, TO(\phi_{p,0})$

$\uparrow$

same orbital integrals appear in Artin-Selberg trace formula