GALOIS REPRESENTATIONS AND UNITARY GROUPS

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1. The goal and strategy

Let F be a totally real field, and E/F be a totally imaginary extension. Let $c \in \mathrm{Gal}(E/F)$ denote the unique non-trivial element, which we call "complex conjugation". We want to explain the ideas behind (some cases of) the following theorem.

Theorem 1.1. *Let $n \geq 2$. If*

(i) π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$,
(ii) $\pi^\vee = \pi$ (self-dual), and
(iii) π_∞ is regular and algebraic,

then for each prime ℓ and $\iota_\ell : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, there exists a semisimple continuous representation

$$\rho_{\ell,\iota_\ell}(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

such that

(a) $\rho_{\ell,\iota_\ell}(\pi)_v$ is unramified for all but finitely many places v of F,
(b) $\rho_{\ell,\iota_\ell}(\pi)_v$ is de Rham at the places $v \mid \ell$, and
(c) (local-global compatibility) for every finite place v of F,

$$\mathrm{WD}(\rho_{\ell,\iota_\ell}(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{\mathrm{F\text{-}ss}} \cong \iota_\ell^{-1}\,\mathrm{LLC}_v(\pi_v).$$

Theorem 1.2 (Theorem 1.3 [Shi09])**.** *Let $n \geq 2$. If*

(i) Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$,
(ii) $\Pi^\vee = \Pi \circ c$ (conjugate self-dual), and
(iii) Π_∞ is regular and algebraic,

then for each prime ℓ and $\iota_\ell : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, there exists a semisimple continuous representation

$$\rho_{\ell,\iota_\ell}(\Pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

such that

(a) $\rho_{\ell,\iota_\ell}(\pi)_w$ is unramified for all but finitely many places w of E,
(b) $\rho_{\ell,\iota_\ell}(\pi)_w$ is de Rham at the places $w \mid \ell$, and
(c) (local-global compatibility) for every finite place w of E,

$$\mathrm{WD}(\rho_{\ell,\iota_\ell}(\Pi)|_{\mathrm{Gal}(\overline{E}_w/E_w)})^{\mathrm{F\text{-}ss}} \cong \iota_\ell^{-1}\,\mathrm{LLC}_v(\Pi_w).$$

Theorem 1.1 can be deduced from Theorem 1.2 via the following pipeline:

$$\pi \xrightarrow{\text{base change}} \Pi \xrightarrow{\text{Theorem 1.2}} \rho_{\ell,\iota_\ell}(\Pi) \xrightarrow{\text{patching}} \rho_{\ell,\iota_\ell}(\pi).$$

We sketch the ideas behind the proof (of some cases) of Theorem 1.2 first, and then we describe how to reduce from Theorem 1.1 to Theorem 1.2 at the end. We do not attempt to prove anything in these notes, but we do try to highlight some key ideas.

Remark 1.3. *In Theorem 1.1 and Theorem 1.2, we have opted to work with*

(i) regular algebraic self-dual cuspidal (RASDC), and
(ii) regular algebraic conjugate self-dual cuspidal (RACSDC)

automorphic representations. More generally, we can work with

(a) regular algebraic essentially self-dual cuspidal (RAESDC), and
(b) regular algebraic essentially conjugate self-dual cuspidal (RAECSDC)

automorphic representations (see [BLGHT11] *for the definition). We should also be able to attach compatible systems of ℓ-adic Galois representations to RAESDC and RAECSDC automorphic representations. This might involve a bit more work.*

The proof of Theorem 1.2 can be conceptualized as the following pipeline:

$$\Pi \text{ on } \mathrm{GL}_n(\mathbb{A}_E) \xrightarrow{\text{descent}} \pi_1 \text{ on } U_1(n) \xrightarrow{\text{Jacquet-Langlands}} \pi_2 \text{ on } U_2(n) \xrightarrow{\text{cohomology}} \rho_{\ell,\iota_\ell}(\Pi)$$

where

(i) $U_1(n)$ is the quasi-split unitary group in n variables defined over F; that is, $U_1(n)$ is quasi-split at all places, including the infinite places where:
$$U_1(n)_\infty \cong U(\lfloor \tfrac{n}{2} \rfloor, \lceil \tfrac{n}{2} \rceil)^{[F:\mathbb{Q}]}.$$
(ii) $U_2(n)$ is a unitary group in n variables defined over F that is quasi-split at all of the finite places, but where at the infinite places we have
$$U_2(n)_\infty \cong U(n-1,1) \times U(n)^{[F:\mathbb{Q}]-1}.$$

The cohomology of the Shimura variety attached to $U_2(n)$ has the right dimension for us to be looking for Galois representations in its cohomology. We elaborate on this later.

2. Classification of unitary groups

Let E/F be a (separable) quadratic algebra with Galois group $\{1, c\}$, where c is the non-trivial automorphism. Let V be a free E-module of rank n.

Definition 2.1. *A Hermitian form on V is a non-degenerate pairing $h : V \times V \to E$ which satisfies the following relations for all $v, w \in V$:*

(i) $h(\alpha v, \beta w) = \alpha \cdot c(\beta) h(v, w)$ and
(ii) $h(v, w) = c(h(w, v))$.

Definition 2.2. *Let (V, h) be a Hermitian space over E. The unitary group $U(V)$ is the subgroup of $\mathrm{GL}(V)$ that preserves h, that is, $U(V) := \{g \in \mathrm{GL}(V) : h(gv, gw) = h(v, w)\}$. The associated algebraic group defined over F has functor of points given by:*
$$U(V)(R) := \{g \in \mathrm{GL}_{E\otimes_F R}(V \otimes_F R) : h(gv, gw) = h(v, w) \text{ for all } v, w \in V \otimes_F R\}.$$
If E is a field (i.e. $E \neq F \times F$), then we say that $U(V)$ is a true unitary group.

We eventually want to classify unitary groups over a totally real number field. First, we give a classification of unitary groups over a characteristic zero local field. We refer the reader to [Har07a] and [Bel09] for more details.

(i) If $E = \mathbb{C}$ and $F = \mathbb{R}$, then any n-dimensional Hermitian space over $\mathbb{C}$ is isomorphic to one of the form $V^{p,q} := (V^+)^p \oplus (V^-)^q$ with $p + q = n$, where
   (a) $V^+ \cong \mathbb{C}$ has Hermitian form $h^+(z, w) = z\overline{w}$.
   (b) $V^- \cong \mathbb{C}$ has Hermitian form $h^-(z, w) = -z\overline{w}$.

Let $U(p,q) := U(V^{p,q})$ be the associated unitary group over $\mathbb{R}$. Here are some facts.
(a) $U(p,q) \cong U(q,p)$ are the only isomorphisms.
(b) $U(n) := U(n,0) = U(0,n)$ is compact; the rest are not.
(c) $U(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ is the unique quasi-split Lie group.

(ii) Let E/F be a quadratic extension of p-adic fields. For each n, there is a list of two n-dimensional Hermitian spaces over E, call them V^+ and V^-. If n is even, then $U(V^+)$ and $U(V^-)$ are not isomorphic. Only one of them, say $U(V^+)$, is quasi-split in the sense that it contains an F-rational Borel subgroup. If n is odd, then $U(V^+) \cong U(V^-)$.

(iii) If $E = F \times F$ is a product of p-adic fields, then for each n, there is a unique unitary group over F associated to an n-dimensional Hermitian space over E, which is GL_n / F. (This is not a true unitary group.)

We attach Hasse invariants (ϵ) to these local unitary groups.

(i) Let $E = \mathbb{C}$ and $F = \mathbb{R}$.

$$\epsilon(U(p,q)) := \begin{cases} 1 & \text{if } p+q \text{ is odd,} \\ (-1)^{m-p} & \text{if } p+q = 2m \text{ is even.} \end{cases}$$

(ii) Let E/F be a quadratic extension of p-adic fields. Set $\epsilon(U(V^+)) = 1$ always.

$$\epsilon(U(V^-)) := \begin{cases} 1 & \text{if } n \text{ is odd,} \\ -1 & \text{if } n \text{ is even.} \end{cases}$$

(iii) Let $E = F \times F$ be the product of p-adic fields. Set $\epsilon(\mathrm{GL}_n / F) = 1$ always.

Let E be a CM field, and F be its maximal totally real subfield. We can now state the classification of unitary groups defined with respect to E/F.

Theorem 2.3. *For each place v of F that is not split in E, choose a sign ϵ_v. For all but finitely many v, including those which split in E, set $\epsilon_v = 1$. Then there exists a global unitary group $G = U(V)$ defined over F if and only if $\prod_v \epsilon_v = 1$. In particular, if n is odd, then any collection of local unitary groups can be realized as a global unitary group.*

3. Descent to a unitary group

Let E be a CM field, and F be its maximal totally real subfield.

Theorem 3.1 (Theorem 3.1.2 [HL04])**.**

Let Π be a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that Π_w, Π_{w^c} are supercuspidal for at least one finite place $w \neq w^c$ of E. Then Π descends to $U_1(n)$; that is, Π is the weak base change of a regular algebraic cuspidal automorphic representation π_1 for $U_1(n)(\mathbb{A}_F)$.

4. Jacquet-Langlands transfer between unitary groups

General Principle 4.1 (p. 5 [Har07a], p. 10 [Har07b])**.**

(i) The only obstruction to the transfer of L-packets between inner forms is local.
(ii) Let K be a local field. Let G, H be inner forms of a reductive group over K. If G is quasi-split, then there are no local obstructions to transfer L-packets from H to G.
(iii) Let G, H be inner forms of a reductive group over $\mathbb{R}$. If G admits a discrete series, in which case so does H, then there are no local obstructions to the transfer of discrete series L-packets from H to G.

The current setup is that we have an automorphic representation π_1 on $U_1(n)(\mathbb{A}_F)$ that we want to transfer to any unitary group U over F such that

$$U_\infty \cong U(n-1,1) \times U(n)^{[F:\mathbb{Q}]-1}.$$

General principle (i) tells us that, after picking such a U, the only obstructions to making this transfer will be local. Since $(\pi_1)_\infty$ is discrete series, general principle (iii) tells us that there are no local obstructions at ∞ to making this transfer for any choice of U. General principle (ii) finally tells us that we should be safe if we can choose that U is quasi-split at all of the finite places. So we should look for a unitary group U such that:

(i) $U_\infty \cong U(n-1,1) \times U(n)^{[F:\mathbb{Q}]-1}$
(ii) U_v is quasi-split at all places v of F

The classification of global unitary groups tells us that this is possible when:

(i) n is odd, or
(ii) $n \equiv 2 \pmod 4$ and $[F : \mathbb{Q}]$ is odd.

Of course, all of these are just vague heuristics, until they are proved for each instance that we are interested in. Let me state some concrete results.

Theorem 4.2 (Theorem 3.1.6 [HL04]).
Let S_0 be a set of finite places of F split in E. Assume $|S_0| \geq 2$. Let U, U' be two inner forms of $U_1(n)$. Let S be a set of places containing S_0 such that:

$$U(\mathbb{A}_F^S) \cong U'(\mathbb{A}_F^S).$$

Let π be a cuspidal automorphic representation of U such that

(i) π_∞ *is regular algebraic*
(ii) π_{S^∞} *is a supercuspidal representation of* U_{S^∞} *(i.e.* π_v *is supercuspidal for all* $v \in S^\infty$*)*
(iii) $\mathrm{JL}_v(\pi_v)$ *is supercuspidal for all* $v \in S_0$

Then there exists a cuspidal automorphic representation π' of U' such that for all $v \notin S$:

$$\pi_v \cong \pi'_v.$$

For us, we can apply the theorem to $U := U_1(n)$ and $U' := U_2(n)$. We remark that this more stringent condition that $\mathrm{JL}_v(\pi_v)$ is supercuspidal implies that we are given more freedom to choose our $U_2(n)$ with the fixed choice of $U_2(n)_\infty$. In particular, we can allow $U_2(n)$ to be non-quasi-split at some of the finite places, which would allow us to treat more cases of when n is even, rather than just the $n \equiv 2 \pmod 4$ and $[F : \mathbb{Q}] \equiv 1 \pmod 2$ case.

5. Cohomology of a unitary Shimura variety

Theorem 5.1 (Theorem 1.5 [Har07b]).
Let Π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Suppose either n is odd, or $[F : \mathbb{Q}]$ is odd and $n \equiv 2 \pmod 4$. Then $\rho_{\ell,\iota_\ell}(\Pi)$ can be realized (up to dualizing and twisting by characters) in $H^{n-1}(\mathrm{Sh}(U_2(n)), L(\Pi_\infty)^\vee_\ell)$, where $\mathrm{Sh}(U_2(n))$ is the Shimura variety attached to the unitary group $U_2(n)$.

Let $G := \mathrm{Res}_{F/\mathbb{Q}} U_2(n)$ be a reductive group over $\mathbb{Q}$.

$$G(\mathbb{R})^+/A_\infty^+ K_\infty^+ = \frac{U(n-1,1) \times U(n)^{[F:\mathbb{Q}]-1}}{U(n-1) \times U(1) \times U(n)^{[F:\mathbb{Q}]-1}} \cong \frac{U(n-1,1)}{U(n-1) \times U(1)}.$$

This space has $\mathbb{R}$-dimension:

$$n^2 - ((n-1)^2 + 1) = 2(n-1).$$

So its $\mathbb{C}$-dimension is:

$$n - 1.$$

So the Shimura variety $\mathrm{Sh}(U_2(n))$ over E has dimension $n-1$. This is compatible with what we are looking for in two ways:

(i) By looking in H^{n-1}, we are looking in the middle degree of cohomology, which is where we should expect to find the automorphic representation π_2 on $U_2(n)$, which is cuspidal and discrete series at infinity.

(ii) For a modular form f, let π_f denote the associated automorphic representation. If we choose a modular curve X of level N_f, then we saw that

$$H^1(X, \overline{\mathbb{Q}}_\ell)_{\pi_f} = H^{0,1} \oplus H^{1,0} = \rho_{\pi_f}$$

with one dimension coming from each of $H^{0,1}$ and $H^{1,0}$. In our case, if the weight of π_2 is such that $L((\pi_2)_\infty)_\ell^\vee \cong \overline{\mathbb{Q}}_\ell$, and if we choose a Shimura variety $\mathrm{Sh}(U_2(n))$ of the right level corresponding to π_2, then we can expect that

$$H^{n-1}(\mathrm{Sh}(U_2(n)), \overline{\mathbb{Q}}_\ell)_{\pi_2} = H^{0,n-1} \oplus \cdots \oplus H^{n-1,0} \cong \rho_{\pi_2}$$

with one dimension coming from each $H^{i,j}$.

6. The reduction: base change and patching

Let F be a fixed number field. Let $\mathcal{I} \neq \varnothing$ be a set of cyclic Galois extension E/F, of prime degree q_E. We allow the prime q_E to vary with $E \in \mathcal{I}$. For every $E \in \mathcal{I}$, we assume we are given an n-dimensional continuous semisimple ℓ-adic Galois representation

$$\rho_E : \Gamma_E := \mathrm{Gal}(\overline{\mathbb{Q}}/E) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

Here ℓ is a fixed prime. The family of representations $\{\rho_E\}$ is assumed to satisfy:

(a) Galois-invariance: $\rho_E^\sigma \cong \rho_E$ for all $\sigma \in \mathrm{Gal}(E/F)$,
(b) Compatibility: $\rho_E|_{\Gamma_{EE'}} \cong \rho_{E'}|_{\Gamma_{EE'}}$ for any $E, E' \in \mathcal{I}$.

These conditions are certainly necessary for ρ_E to be of the form $\rho|_{\Gamma_E}$ for some ρ of Γ_F. In fact, (a) and (b) are also sufficient conditions [Sor20, Lemma 2] if the collection $\mathcal{I}$ is large enough in the following sense.

Definition 6.1 (Definition 1 [Sor20])**.**
For a finite set S of places of F, we say that a non-empty collection $\mathcal{I}$ of cyclic extensions E/F of prime degree is S-general if for any finite place $v \notin S$ of F there exist infinitely many $E \in \mathcal{I}$ in which v splits completely.

Example 6.2. *The family of imaginary quadratic extensions*

$$\mathcal{I} = \{\mathbb{Q}(\sqrt{-p})/\mathbb{Q} : p \text{ prime}\}$$

over $\mathbb{Q}$ is $\varnothing$-general. If F is a totally real field, then

$$F\mathcal{I}$$

is a $\varnothing$-general family of CM extensions of F.

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