Galois representations of Hilbert modular forms: an overview.

Recall. $G = GL_{2,\mathbb{Q}}$

f modular form of wt 2 $\rightsquigarrow$ $\rho_f := H^1_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_f \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

X = modular curve $/\mathbb{Q}$

$X(\mathbb{C}) = \Gamma \backslash \mathcal{H}^*$

- 2-dim'l b/c $H^1(X(\mathbb{C}), \mathbb{C})_f \cong \underbrace{\mathbb{C}\omega_f}_{H^{1,0}} \oplus \underbrace{\mathbb{C}\overline{\omega_f}}_{H^{0,1}}$
- $a_p(f) = \mathrm{Tr}(\mathrm{Frob}_p \circlearrowright \rho_f)$

  Eichler–Shimura relationship

Today. $G = GL_{2,F}$, $[F:\mathbb{Q}] = d$, totally real, $\sigma_i : F \hookrightarrow \mathbb{R}$, $i = 1, \dots, d$

$$\Gamma \leq SL_2(\mathcal{O}_F) \hookrightarrow SL_2(\mathbb{R})^d \circlearrowright \mathcal{H}^d$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \left(\begin{bmatrix} \sigma_i(a) & \sigma_i(b) \\ \sigma_i(c) & \sigma_i(d) \end{bmatrix} \right)_{i=1,\dots,d}$$

Def. Hilbert modular form of weight $\underline{k} = (k_1, \dots, k_d)$ s.t. $k_1 \equiv \dots \equiv k_d \bmod 2$, level $\Gamma \leq SL_2(\mathcal{O}_F)$:

$f : \mathcal{H}^d \longrightarrow \mathbb{C}$ holomorphic, $f|_{\underline{k}}\gamma = f \;\; \forall \gamma \in \Gamma$.

Thm (Ohta, Rogawski–Tunnell, Carayol, Wiles, Taylor, Blasius–Rogawski).

Given f newform level $\mathfrak{n}$ with coeff. in L, ℓ prime, $\lambda | \ell$ in L,

$\exists\, \rho_{f,\lambda} : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(L_\lambda)$ s.t.

$\forall\, \mathfrak{p}$ prime to ℓN, $\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}} \circlearrowright \rho_{f,\lambda}) = a_{\mathfrak{p}}(f)$.

Goal. Explain how to (& how not to) prove this theorem!

Analogue of modular curve:

$$\Gamma \backslash \mathcal{H}^d = \text{complex manifold of dim } d$$

$\rightsquigarrow \exists\, X =$ Hilbert modular variety, dim d / $\mathbb{Q}$
s.t. $X(\mathbb{C}) \supseteq \Gamma \backslash \mathcal{H}^d$ (compactification).

Obvious guess: find $\rho_{f,\lambda}$ in $H^d_{et}(X_{\overline{\mathbb{Q}}}, L_\lambda)$, $\underline{k} = (2, \ldots, 2)$.

<u>Cohomology of X:</u> $d = 2$ for simplicity, $\underline{k} = (2,2)$

$$f \;\begin{cases} \omega_f = f(z_1, z_2)\, dz_1 \wedge dz_1 & \in H^{2,0} \\ & \oplus \\ \eta_f^1 = f(-\overline{z}_1, z_2)\, d\overline{z}_1 \wedge dz_2 & \oplus \\ & H^{1,1} \\ \eta_f^2 = f(z_1, -\overline{z}_2)\, dz_1 \wedge d\overline{z}_2 & \in \\ & \oplus \\ \overline{\omega_f} = f(-\overline{z}_1, -\overline{z}_2)\, d\overline{z}_1 \wedge d\overline{z}_2 & \in H^{0,2} \end{cases} \cong H^2(\Gamma \backslash \mathcal{H}^2, \mathbb{C})$$

(Hodge decomposition)

$\Rightarrow$ 4-dimensional!

<u>In general:</u> $H^d(X(\mathbb{C}), \mathbb{C})_f = 2^d$-dimensional

$$\Rightarrow H^d_{et}(X_{\overline{\mathbb{Q}}}, L_\lambda)_f \rightsquigarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_{2^d}(L_\lambda)$$

instead of: $\mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(L_\lambda)$ ...

(Also, no chance to get an elliptic curve, since
X is not a curve $\Rightarrow$ "$\mathrm{Jac}(X)$" makes no sense ...)

<u>Rmk.</u> $H^1_{et}(X_{\overline{\mathbb{Q}}}, L_\lambda)_f = 0 \Rightarrow$ can't find it there either...

Ideas to fix this :

1. <u>Carayol</u> : Jacquet–Langlands transfer $Gl_{2,F} \rightsquigarrow B^\times$ for B = quat. algebra / F.
(Rogawski–Tunnell, Ohta)

under assumption (*) (introduced soon).

2. <u>Blasius–Rogawski</u> : $GL_{2,\mathbb{R}}$ is closely related to $\underline{U(1,1)}$
$\Rightarrow$ endoscopic transfer to $\underline{U(2,1)}$ & construct Galois rep there!

<u>Key point</u>: They consider X = Picard modular surface for G = unitary group / F.

$$\rho_{f,\lambda} := \underbrace{H^1_{ét}(X_{\bar F}, \bar{\mathbb{Q}}_\ell)_\pi}_{\text{2-dim'l } (\neq 0)}, \quad \pi = \text{transfer of } f \text{ to } G.$$

Removes (*) but need to assume $\underline{k} \neq (2, \dots, 2)$.

3. <u>Wiles</u>: use families of HMFs to reduce to cases covered by 1. ; replace (*) with "ordinary" assumption.

<u>Taylor</u>: remove ordinary assumption $\Rightarrow$ cover all cases!

Today, we do 1. !

1.1. Local Jacquet-Langlands.

F = local field (finite ext. of $\mathbb{R}$ or $\mathbb{Q}_p$)

Fact. $\exists$ two simple rank 4 F-algebras with center F (QA)

$M_2(F)$ & D = division algebra.

"split" "non-split"

E.g. over $\mathbb{R}$: $M_2(\mathbb{R})$ & $\mathbb{H}$ = Hamilton quaternions.

Jacquet-Langlands correspondence:

$$\left\{\begin{array}{c}\text{smooth irred.}\\ \text{reps of } D^\times\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{smooth irred.}\\ \text{reps of } GL_2(F)\end{array}\right\}$$

$F=\mathbb{R}$.

- $\chi \circ \nu : \mathbb{H}^\times \longrightarrow \mathbb{C}^\times$ for $\chi : \mathbb{R}^\times \longrightarrow \mathbb{R}^\times$, $\nu : \mathbb{H}^\times \longrightarrow \mathbb{R}^\times$ reduced norm $\longmapsto$ $D_2 \otimes \chi$, D_2 = discrete series of "wt 2"
- k-dim'l rep. $\longmapsto$ $D_k \otimes \chi$, D_k = discrete series of "wt $k+1$"

discrete series

$F/\mathbb{Q}_p$

- $\chi \circ \nu : D^\times \longrightarrow \mathbb{C}^\times$ for $\chi : F^\times \longrightarrow \mathbb{C}^\times$, $\nu : D^\times \longrightarrow F^\times$ reduced norm $\longmapsto$ $St \otimes \chi$
- $\dim > 1$ $\longmapsto$ supercuspidal

"discrete series"

Rmk. For $B \in \{M_2(F), D\}$, will consider $JL : \mathrm{Rep}(B^\times) \longrightarrow \mathrm{Rep}(GL_2)$; identity when $B = M_2(F)$.

1.2. Global Jacquet-Langlands.

F = totally real degree $d/\mathbb{Q}$

B = quat. algebra $/F$ $\rightsquigarrow$ v place of F, $B_v := B \otimes_F F_v$

$$\varepsilon(B_v) = \begin{cases} +1 & B_v = M_2(F_v) \text{ split} \\ -1 & B_v = D_v \text{ non-split} \end{cases}$$

$$\{B = QA/F\} \overset{\text{'1-1'}}{\longleftrightarrow} \left\{ \begin{array}{l} \varepsilon(B_v) = +1 \text{ for a.a. } v \\ \& \prod_v \varepsilon(B_v) = +1 \text{ (even \# of } -1\text{'s)} \end{array} \right\}$$

Jacquet-Langlands correspondence.

$$\left\{ \begin{array}{l} \text{aut. reps} \\ \text{of } B^\times(\mathbb{A}) \end{array} \right\} \xrightarrow{JL} \left\{ \begin{array}{l} \text{aut. reps} \\ \text{of } GL_2(\mathbb{A}) \end{array} \right\}$$

$$\pi' = \bigotimes_v \pi'_v \longmapsto \bigotimes_v JL_v(\pi'_v)$$

Proof: trace formula.

Def. f = Hilbert mod. form $\rightsquigarrow$ π = aut. rep. of f

f transfers to $B^\times$ if $\pi \cong \bigotimes_v JL_v(\pi_v^B)$ for π^B = aut. rep. of $B^\times$

Explicitly: $B_v = M_2(F_v)$: no condition

$B_v = D_v$: π_v = discrete series

$\mathcal{O}_B \subseteq B$ max'l order $\rightsquigarrow$ $\Gamma \leq \mathcal{O}_B^\times$ level structure

$(M_2(\mathcal{O}_F) \subseteq M_2(F))$ $\qquad$ $(\Gamma_0(\mathfrak{n}) \subseteq SL_2(\mathcal{O}_F))$

$\Sigma_\infty(B) = \{v | \infty : B_v = M_2(\mathbb{R})\}$ $\rightsquigarrow$ $\mathcal{H}_B := \mathcal{H}^{\Sigma_\infty(B)} \times \mathbb{H}^{\Sigma_\infty \setminus \Sigma_\infty(B)}$

$\subseteq \Sigma_\infty = \{v | \infty \text{ in } F\}$ $\qquad$ $\circlearrowleft \Gamma$

and $\Gamma \backslash \mathcal{H}_B$ = complex manifold of dim $|\Sigma_\infty(B)|$,

compact if $B \neq GL_{2,F}$.

Thm. $G_B := \{ \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma(\Sigma_\infty(B)) = \Sigma_\infty(B) \} \supseteq Gal(\overline{\mathbb{Q}}/F)$

$\rightsquigarrow F_B = \overline{F}^{G_B} \subseteq F$

$\Rightarrow {}_{\Gamma}\backslash\mathcal{H}_B = X_B^\Gamma(\mathbb{C})$ for $X_B^\Gamma = |\Sigma_\infty(B)|$ – dim'l variety $/F_B$.

E.g. • $B^\times = GL_{2,F} \Rightarrow$ recover Hilbert modular variety

• $|\Sigma_\infty(B)| = d-1 \Rightarrow X_B^\Gamma$ is a curve $/F$.

Thm (Carayol). Suppose

(*) f transfers to some B s.t. $|\Sigma_\infty(B)| = 1$.

Explicitly : • d is odd ($\Rightarrow$ take $\varepsilon_v(B) = -1$ at $d-1$ ∞-places v)

($k_i \geq 2\ \forall i$) • d is even & π_w = d.s. at some $w \nmid \infty$

($\Rightarrow$ take $\varepsilon_w(B) = -1$ & $\varepsilon_v(B) = -1$ at $d-1$ ∞-places v)

Then $\exists$ curve X_B^Γ / F & can repeat the Eichler–Shimura construction using this curve!

In particular : $H^1_{et}((X_B^\Gamma)_{\overline{F}}, L_\lambda)_{\pi^B} \circlearrowleft Gal(\overline{F}/F)$ is the 2-dim'l Galois rep. $\rho_{f,\lambda}$ we wanted.

The difficulty is : X_B^Γ has no good moduli interpretation

$\Rightarrow$ the story is harder.

How to remove the assumption (*) ?

$\longrightarrow$ Wiles / Taylor : use "congruences" to reduce to the above.

$\longrightarrow$ Blasius – Rogawski : use a different instance of functoriality.

2. Bonus.

What was the rep.

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_{\mathbb{Q}}} GL_{2^d}(L_\lambda) \; ?$$

Thm (Langlands, Brylinski – Labesse). $\rho_{\mathbb{Q}} \cong \otimes\text{-}\mathrm{Ind}_F^{\mathbb{Q}} \, \rho_{f,\lambda}$.

$H \leq G$ finite index $H \circlearrowright W \rightsquigarrow \otimes\text{-}\mathrm{Ind}\, W := \bigotimes_{\sigma H \in G/H} \sigma W \circlearrowleft G$

Langlands: $B \neq M_2(F)$ s.t. $\Sigma_\infty(B) = \emptyset$

Brylinski – Labesse: deal with $B = M_2(F)$, i.e. compactifications.