WHAT IS A REGULAR ALGEBRAIC AUTOMORPHIC REPRESENTATION?

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1. Introduction

Let F be a number field. Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

Definition 1. We say that π is *regular algebraic* if π_∞ has the same infinitesimal character as an irreducible algebraic representation W of $(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n)_\mathbb{C}$.

The purpose of these notes is to demystify this definition. For example, why do we consider representations of $(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n)_\mathbb{C}$ rather than, say, $(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n)_\mathbb{R}$? The latter satisfies

$$(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n)(\mathbb{R}) = \mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{R}) = \mathrm{GL}_n(\prod_{v|\infty} F_v) = \mathrm{GL}_n(\mathbb{A}_{F,\infty}).$$

We will go through one example in detail, which should clarify some things.

Let T_n be the standard diagonal torus and B_n be the standard upper triangular Borel subgroup in GL_n. Identify $X^\bullet(T_n)$ with $\mathbb{Z}^n$ in the usual way, and write $\mathbb{Z}^n_+ \subset \mathbb{Z}^n$ for the subset of weights which are B_n-dominant, that is:

$$\mathbb{Z}^n_+ = \{(a_1, \dots, a_n) \in \mathbb{Z}^n : a_1 \geq \dots \geq a_n\}.$$

There is a bijection between irreducible (finite-dimensional) algebraic representations of GL_n (over any characteristic zero field) and $\mathbb{Z}^n_+$, by sending the representation to its highest weight.

The irreducible representations of $(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n)_\mathbb{C}$ are then easy to determine. Indeed,

$$(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n)(\mathbb{C}) = \mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{C}) = \mathrm{GL}_n(\prod_{\mathrm{Hom}(F,\mathbb{C})} \mathbb{C}) = \prod_{\mathrm{Hom}(F,\mathbb{C})} \mathrm{GL}_n(\mathbb{C}).$$

So its irreducible representations are in bijection with the set $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\mathbb{C})}$.

2. Example

Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field. We will be considering automorphic representations of $\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times$, that is, Hecke characters. (The first part of our exposition closely follows that of [Sno10], before breaking off to do our own calculations.)

To this end, let $\psi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a Hecke character. We consider

$$\psi_\infty : (F \otimes_\mathbb{Q} \mathbb{R})^\times \to \mathbb{C}^\times.$$

Fix roots of the integer d: $u \in F$ and $u' \in \mathbb{C}$; this determines a unique embedding $F \xrightarrow{\sigma} \mathbb{C}$ which sends u to u'. In fact, σ induces an isomorphism $F \otimes_\mathbb{Q} \mathbb{R} \to \mathbb{C}$ sending $x \otimes y \mapsto \sigma(x)y$. Therefore, ψ_∞ factors through this isomorphism

$$\psi_\infty : (F \otimes_\mathbb{Q} \mathbb{R})^\times \to \mathbb{C}^\times \to \mathbb{C}^\times.$$

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Recall: every continuous homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$ has the form

$$re^{i\theta} \mapsto r^a e^{in\theta}$$

for some $a \in \mathbb{C}$ and $n \in \mathbb{Z}$ (this is because $\mathbb{C}^\times = \mathbb{R}_{>0} \times \mathbb{S}^1$ where $\mathbb{S}^1$ is the unit circle). Let $z := x + yu'$ with $x, y \in \mathbb{R}$ and $xy \neq 0$ be an arbitrary element of $\mathbb{C}^\times$; note that this is the image of $1 \otimes x + u \otimes y$ under the isomorphism $F \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}$. In polar coordinates,

$$z = (x^2 - dy^2)^{1/2} \frac{x + yu'}{(x^2 - dy^2)^{1/2}}.$$

Then z is mapped under $\mathbb{C}^\times \to \mathbb{C}^\times$ to the following element:

$$(x^2 - dy^2)^{a/2} \left(\frac{x + yu'}{(x^2 - dy^2)^{1/2}} \right)^n = (x^2 - dy^2)^{(a-n)/2} (x + yu')^n.$$

This is a rational function of x and y if and only if $a - n$ is an even integer, say $2m$. (We want to treat ψ_∞ as a function of x and y, rather than as a function of z, because $\mathrm{GL}_n(\mathbb{A}_{F,\infty}) = (\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n)(\mathbb{R})$ always has the structure of a real algebraic group, and may not have complex structure if the base field F, say, is totally real.) It follows that ψ_∞, viewed as a function on $\mathbb{C}^\times$, takes the following form:

$$z = x + yu' \mapsto (x + yu')^{m+n} (x - yu')^m = z^{m+n} \overline{z}^m.$$

So if ψ_∞ is an algebraic character of $(F \otimes_{\mathbb{Q}} \mathbb{R})^\times$, it follows that there exist integers m and n such that after identifying $(F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ with $\mathbb{C}^\times$, the character ψ_∞ looks like:

$$z \mapsto z^m \overline{z}^n.$$

Let's do our own calculation now: what is the infinitesimal character of ψ_∞? First, we need to take the derivative $d\psi_\infty$ to obtain an action of the real Lie algebra

$$\mathrm{Lie}((F \otimes_{\mathbb{Q}} \mathbb{R})^\times) = F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}(1 \otimes 1) \oplus \mathbb{R}(u \otimes 1) \xrightarrow{\sim} \mathbb{R} \oplus \mathbb{R}u' = \mathbb{C}.$$

Let's compute $d\psi_\infty$ on the basis $\{1, u'\}$. Fo any $\gamma \in \mathbb{C}$:

$$d\psi_\infty(1)(\gamma) = \frac{d}{dt}\bigg|_{t=0} e^t \cdot \gamma = \frac{d}{dt}\bigg|_{t=0} e^{t(m+n)} \gamma = (m+n)\gamma$$

$$d\psi_\infty(u')(\gamma) = \frac{d}{dt}\bigg|_{t=0} e^{tu'} \cdot \gamma = \frac{d}{dt}\bigg|_{t=0} e^{tu'(m-n)} \gamma = u'(m-n)\gamma$$

So the $\mathbb{R}$-linear homomorphism $d\psi_\infty : \mathbb{C} \to \mathbb{C}$ satisfies

$$d\psi_\infty(1) = m + n$$
$$d\psi_\infty(u') = u'(m - n)$$

Note that at this stage, m and n are still somewhat entangled. Let us recall for a moment what we are after. If $\mathfrak{g} := \mathrm{Lie}((F \otimes_{\mathbb{Q}} \mathbb{R})^\times)$, then we want to compute the infinitesimal central character of ψ_∞, which is a character of $Z(U(\mathfrak{g}_{\mathbb{C}}))$: the centre of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. In particular, we need to compute $\mathfrak{g}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$; this is where the magic happens, and things will become untangled.

Recall there is a canonical $\mathbb{C}$-linear isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$
$$z \otimes w \mapsto (zw, z\overline{w})$$

where $\mathbb{C}$ acts on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ in the first factor, and on $\mathbb{C} \times \mathbb{C}$ diagonally. This is a special case of the following more general phenomenon; for details, please read [Gai].

Proposition 2. *Let L/K be a finite Galois extension, with Galois group G, and let A be an L-algebra. For each $\sigma \in G$, let $*_\sigma$ denote the twisted scalar action of L on A by $z *_\sigma a = \sigma(z)a$. Let A_σ denote the resulting L-algebra. Then there is an L-algebra isomorphism*

$$L \otimes_K A \to \prod_{\sigma \in G} A_\sigma$$
$$z \otimes a \mapsto (z *_\sigma a)_{\sigma \in G}.$$

In our case, choose $A = L = \mathbb{C}$ and $K = \mathbb{R}$, and let $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denote the unique non-trivial element. Then there is a $\mathbb{C}$-algebra isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}_c$$
$$z \otimes w \mapsto (zw, \overline{z}w).$$

However, (coincidentally) there is a canonical $\mathbb{C}$-algebra isomorphism

$$\mathbb{C} \to \mathbb{C}_c$$
$$w \mapsto \overline{w}.$$

The composition of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}_c$ and $\mathbb{C} \times \mathbb{C}_c \to \mathbb{C} \times \mathbb{C}$ gives us the desired $\mathbb{C}$-algebra isomorphism alluded to in the beginning. Identifying $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ with $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, we see that a natural basis for $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ gets mapped under the isomorphism to the following elements:

$$1 \otimes 1 \mapsto (1, 1)$$
$$1 \otimes u' \mapsto (1, -u')$$
$$i \otimes 1 \mapsto (i, i)$$
$$i \otimes u' \mapsto (iu', -iu')$$

The next step is to observe that we can automatically upgrade the $\mathbb{R}$-linear homomorphism $d\psi_\infty : \mathfrak{g} \to \mathbb{C}$ to a $\mathbb{C}$-linear homomorphism $(d\psi_\infty)_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. Let's calculate this action on the $\mathbb{C}$-linear basis $(1, 0)$ and $(0, 1)$ of $\mathfrak{g}_{\mathbb{C}}$. Indeed (pardon my abuse of notation),

$$(1, 0) = \tfrac{1}{2}((1 \otimes 1) - \tfrac{iu'}{d}(i \otimes u'))$$
$$(0, 1) = \tfrac{1}{2}((1 \otimes 1) + \tfrac{iu'}{d}(i \otimes u')).$$

The $\mathbb{C}$-linearity of $(d\psi_\infty)_{\mathbb{C}}$ then tells us that:

$$(d\psi_\infty)_{\mathbb{C}}(1, 0) = \tfrac{1}{2}((d\psi_\infty)_{\mathbb{C}}(1 \otimes 1) + \tfrac{u'}{d}(d\psi_\infty)_{\mathbb{C}}(1 \otimes u')) = \tfrac{1}{2}((m + n) + \tfrac{u'}{d}u'(m - n)) = m$$
$$(d\psi_\infty)_{\mathbb{C}}(0, 1) = \tfrac{1}{2}((d\psi_\infty)_{\mathbb{C}}(1 \otimes 1) - \tfrac{u'}{d}(d\psi_\infty)_{\mathbb{C}}(1 \otimes u')) = \tfrac{1}{2}((m + n) - \tfrac{u'}{d}u'(m - n)) = n.$$

So the m and n have been unentangled!

Since $\mathfrak{g}_{\mathbb{C}}$ is abelian, the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to the polynomial algebra $\mathbb{C}[X, Y]$ where $X = (1, 0)$ and $Y = (0, 1)$. Its centre is equal to itself. So the infinitesimal central character of ψ_∞ is the character

$$\mathbb{C}[X, Y] \to \mathbb{C}$$
$$X \mapsto m$$
$$Y \mapsto n.$$

Finally, let's see which algebraic character of $(\operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_1)_\mathbb{C}$ this comes from. Indeed, using the isomorphism $\mathbb{C} \otimes_\mathbb{R} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ from before, we have:

$$(\operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_1)(\mathbb{C}) = (F \otimes_\mathbb{Q} \mathbb{C})^\times = ((F \otimes_\mathbb{Q} \mathbb{R}) \otimes_\mathbb{R} \mathbb{C})^\times = (\mathbb{C} \otimes_\mathbb{R} \mathbb{C})^\times = \mathbb{C}^\times \times \mathbb{C}^\times.$$

All the algebraic characters of $\mathbb{C}^\times$, considered as the $\mathbb{C}$-points of $(\mathbb{G}_m)_\mathbb{C}$ rather than as the $\mathbb{R}$-points of $(\operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_1)_\mathbb{R}$, are given by $z \mapsto z^n$ for some integer n.

With a simple calculation, we easily see that

$$\begin{aligned} \phi : (\operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_1)(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times &\to \mathbb{C}^\times \\ (z, w) &\mapsto z^m w^n \end{aligned}$$

has the same infinitesimal character as ψ_∞; in fact, $d\phi \cong (d\psi_\infty)_\mathbb{C}$. Going back to what we said in the introduction, we see that ϕ is associated to the tuple

$$(m, n) \in \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^{\operatorname{Hom}(F,\mathbb{C})} = (\mathbb{Z}^1_+)^{\operatorname{Hom}(F,\mathbb{C})}.$$

References

[Gai] Pierre-Yves Gaillard. Tensor product algebra $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$. Mathematics Stack Exchange. URL: `https://math.stackexchange.com/q/2265459` (version: 2017-05-05).

[Sno10] Andrew Snowden. Lecture 11: Hecke characters and Galois characters, 2010. URL: `http://math.stanford.edu/~conrad/modseminar/pdf/L11.pdf`. Last edited on 2010/01/28. Last visited on 2024/08/10.