LOCAL ZETA FUNCTIONS AND THE FUNCTIONAL EQUATION

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1. Fourier Analysis

Definition 1. *If G is an abelian locally compact Hausdorff topological group (LCHTG) then the character group $\hat{G} = \mathrm{Hom}(G, \mathbb{S}^1)$ is called the* Pontryagin dual *of G.*

Remark 2. *The term* character *will refer to continuous maps into the unit circle $\mathbb{S}^1 \subset \mathbb{C}^\times$. For arbitrary continuous maps into $\mathbb{C}^\times$, we shall refer to them as* quasi-characters.

Proposition 3. *Let G be an abelian LCHTG, then the canonical map $G \to \hat{\hat{G}}$ is an isomorphism of topological groups.*
[Buz08, p.46]

Definition 4. *If $f \in L^1(G)$, i.e. $f : G \to \mathbb{C}$ is integrable, we define its* Fourier transform *to be the function $\hat{f} : \hat{G} \to \mathbb{C}$ given by the formula*

$$\hat{f}(\lambda) := \int_G f(g)\overline{\lambda(g)}\, dg.$$

Theorem 5. *Fix Haar measures on G and $\hat{G}$. Then there exists a real number $c > 0$ such that if $f \in L^1(G)$ is continuous, and $\hat{f} \in L^1(\hat{G})$, and if we identify $G \cong \hat{\hat{G}}$, then*

$$\hat{\hat{f}}(g) = cf(g^{-1})$$

for all $g \in G$. Moreover, for any choice of Haar measure on G, there exists a unique Haar measure on $\hat{G}$ such that $c = 1$.
[Buz08, p.47]

2. Quasi-Characters of K^+ and $K^\times$

Let K be the completion of a number field at a finite or infinite place v. Let $K^+ \ni \xi$ denote the additive group, and $K^\times \ni \alpha$ denote the multiplicative group.

Lemma 6. *Let $\xi \mapsto \lambda(\xi)$ be a non-trivial character of K^+. Then for each $\eta \in K^+$, $\xi \mapsto \lambda(\eta\xi)$ is also a character. The correspondence $\eta \mapsto (\xi \mapsto \lambda(\eta\xi))$ is an isomorphism of topological groups, identifying $K^+ \cong \hat{K^+}$.*
[CF77, Lemma 2.2.1]

Let $U \subset K^\times$ be the subgroup of α such that $|\alpha| = 1$. It is always compact. If K is non-archimedean, then it is also open.

Definition 7. *A quasi-character $c : K^\times \to \mathbb{C}^\times$ is* unramified *if $c(U) = 1$.*

Lemma 8. *The unramified quasi-characters of $K^\times$ are all of the form $c(\alpha) = |\alpha|^s := e^{s \log|\alpha|}$ for some $s \in \mathbb{C}$. If K is archimedean, then s is determined by c. If K is non-archimedean, then s is only determined modulo $\frac{2\pi i}{\log q}\mathbb{Z}$ where q is the size of the residue field of K.*
[CF77, Lemma 2.3.1], [Buz08, p.51]

Theorem 9. *The quasi-characters of $K^\times$ are all of the form $c(\alpha) = \widetilde{c}(\widetilde{\alpha})|\alpha|^s$ where $\widetilde{c}$ is a character of U uniquely determined by c, and s is determined by the previous lemma.* [CF77, Theorem 2.3.1], [Buz08, p.52]

Proof. We can split the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] & K^\times \arrow[r] & |K^\times| \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0. \end{tikzcd}$$

For K archimedean, define s so that $\operatorname{im} s = \mathbb{R}_{>0}$. For K non-archimedean, define s so that $\operatorname{im} s = \pi^{\mathbb{Z}}$ where $\pi \in K^\times$ is a fixed uniformizer. Write $K^\times = U \times V$ where $V = \operatorname{im} S$. For $\alpha \in K^\times$, let $\widetilde{\alpha}$ denote its projection onto U. Since U is compact, $\widetilde{c} := c|_U$ has image in $\mathbb{S}^1$. Finally, $c/\widetilde{c}$ is an unramified quasi-character of $K^\times$, which we have already classified. ☐

We now equip $\operatorname{Hom}(K^\times, \mathbb{C}^\times)$ with the structure of a 1-dimensional complex manifold.

$$\begin{aligned} \operatorname{Hom}(K^\times, \mathbb{C}^\times) &= \operatorname{Hom}(U, \mathbb{C}^\times) \times \operatorname{Hom}(V, \mathbb{C}^\times) \\ &= \operatorname{Hom}(U, \mathbb{S}^1) \times \operatorname{Hom}(V, \mathbb{C}^\times) \\ &= \hat{U} \times \operatorname{Hom}(V, \mathbb{C}^\times) \end{aligned}$$

The Pontryagin dual of a compact space is discrete, so $\hat{U}$ is discrete. We have already seen that the space $\operatorname{Hom}(V, \mathbb{C}^\times)$ of unramified quasi-characters is parameterized by a single complex number s. If K is archimedean, then this space is isomorphic to $\mathbb{C}$. Otherwise, it is isomorphic to the cyclinder $\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z}$ for some integer q as defined above.

Therefore, $\operatorname{Hom}(K^\times, \mathbb{C}^\times)$ can be given the structure of a disjoint union of 1-dimensional complex manifolds. The connected components correspond to fixing a character of U.

Definition 10. *Two quasi-characters of $K^\times$ are said to be* equivalent *if their quotient is an unramified quasi-character. We remark that two quasi-characters are equivalent if and only if they lie on the same connected component of* $\operatorname{Hom}(K^\times, \mathbb{C}^\times)$.

Example 11. *Let $c : K^\times \to \mathbb{C}^\times$ be a quasi-character.*

(1) $K = \mathbb{R}$, $U = \{\pm 1\}$, $V = \mathbb{R}_{>0}$.

$$c(\alpha) = \operatorname{sgn}(\alpha)^\varepsilon |\alpha|^s, \qquad \varepsilon \in \{0, 1\}, s \in \mathbb{C}.$$

(2) $K = \mathbb{C}$, $U = \mathbb{S}^1$, $V = \mathbb{R}_{>0}$.

$$\hat{\mathbb{S}^1} = \mathbb{Z} = \{n : z \mapsto z^n\}$$

$$c(\alpha) = \widetilde{\alpha}^n |\alpha|^s, \qquad n \in \mathbb{Z}, s \in \mathbb{C}.$$

(3) $K = \mathbb{Q}_p$, $U = \mathbb{Z}_p^\times$, $V = p^{\mathbb{Z}}$.

$$\operatorname{Hom}(\mathbb{Z}_p^\times, \mathbb{C}^\times) = \varinjlim_n \operatorname{Hom}((\mathbb{Z}/p^n\mathbb{Z})^\times, \mathbb{C}^\times)$$

$$c(\alpha) = (\text{Dirichlet character})|\alpha|^s, \qquad s \in \mathbb{C}/\tfrac{2\pi i}{\log p}\mathbb{Z}.$$

Definition 12. *Let $c : K^\times \to \mathbb{C}^\times$ with $c = \widetilde{c}|-|^s$. We call $\sigma := \operatorname{Re}(s)$ the* exponent *of c. We denote the exponent of c by* $\exp(c)$.

Remark 13. *We shall soon see that quasi-characters c with $\exp(c) > \sigma$ will correspond to complex numbers s with $\operatorname{Re} s > 0$ when speaking about the domain of zeta functions.*

3. Local Zeta Functions and the Functional Equation

Let q be the size of the residue field of K. Fix a Haar measure $d\xi$ on K^+. Then we define the Haar measure on $K^\times$ by the formulas

$$d^\times \alpha := \frac{d\alpha}{|\alpha|} \qquad (K \text{ archimedean})$$

$$d^\times \alpha := \frac{q}{q-1}\frac{d\alpha}{|\alpha|} \qquad (K \text{ non-archimedean})$$

We've already identified $K^+ \cong \hat{K^+}$, which allows us to further identify $L^1(K^+) \cong L^1(\hat{K^+})$. Let $f(\xi)$ denote a complex-valued function on K^+, and $f(\alpha)$ denote its restriction to $K^\times$.

Definition 14. *Let $Z = Z(K)$ denote the space of functions $f : K \to \mathbb{C}$ such that*

(1) $f, \hat{f} : K \to \mathbb{C}$ are both continuous, and in $L^1(K^+)$,
(2) $x \mapsto f(x)|x|^\sigma$ and $x \mapsto \hat{f}(x)|x|^\sigma$ are both in $L^1(K^\times)$ for all $\sigma > 0$.

[CF77, p.313]

Definition 15. *Let $S = S(K)$ denote the space of functions $f : K \to \mathbb{C}$ such that*

(1) for K archimedean: f is a smooth function such that

$$p(x)f(x) \to 0$$

as $x \to \infty$ for all polynomials p.

(2) for K non-archimedean: f is a smooth function with compact support.

[RV99, p.245]

I am pretty sure that $S \subset Z$ and this inclusion is dense in the right topologies. But I cannot be bothered to right down a proof. We will start by developing the theory of zeta functions using the larger space of functions Z. This is the "largest" space of functions on which the theory can be made to work, and is what Tate originally considered in his thesis. Later on, we will restrict our attention to S, and this will allow us to make the connection between zeta functions and the local L-functions that one typically writes down.

Definition 16. *For each $f \in Z$, we introduce a function $\zeta(f, c)$ of quasi-characters c, defined for all quasi-characters of exponent $\sigma > 0$, by the formula*

$$\zeta(f, c) := \int_{K^\times} f(\alpha)c(\alpha)\ d^\times\alpha.$$

*This is called a ζ-*function *of K.*

Lemma 17. *The function $\zeta(f, c)$ converges, and is holomorphic in the domain* $\exp(c) > 0$. [CF77, Lemma 2.4.1], [Buz08, p.54]

Proof. Convergence is immediate: since $f \in Z$ we have $|f(x)c(x)| = |f(x)||x|^{\exp(c)} \in L^1(K^\times)$. The complex structure near c is given by $c|-|^s$ for small s. We just need to check that $\zeta(f, c|-|^s)$ is holomorphic in s for small s. Write

$$\zeta(f, c|-|^s) = \int_{K^\times} f(\alpha)c(\alpha)|\alpha|^s\ d^\times\alpha.$$

By our assumptions, differentiating with respect to s commutes with the integral, and so

$$\frac{d}{ds}\int_{K^\times} f(\alpha)c(\alpha)|\alpha|^s \, d^\times\alpha = \int_{K^\times} f(\alpha)c(\alpha)\frac{\partial}{\partial s}|\alpha|^s \, d^\times\alpha = \int_{K^\times} f(\alpha)c(\alpha)|\alpha|^s \log|\alpha| \, d^\times\alpha.$$

We are just left to show that the final integral converges. Fix a small s.

(1) As $|\alpha| \to \infty$, we have $|c(\alpha)|\alpha|^s \log|\alpha|| \leq |\alpha|^{\exp(c)+\mathrm{Re}(s)+\varepsilon}$ for any $\varepsilon > 0$. So the integral converges on the open vertical strip $|\mathrm{Re}(s)| < \varepsilon + \exp(c)$ for any $\varepsilon > 0$.

(2) For $|\alpha| \to 0$, let $\beta = 1/\alpha$ and instead require $|\beta| \to \infty$.

$$\begin{aligned} |c(\alpha)|\alpha|^s \log|\alpha|| &= |\alpha|^{\exp(c)+\mathrm{Re}(s)}|\log|\alpha|| \\ &= |\beta|^{-\exp(c)-\mathrm{Re}(s)}|\log|\beta|| \\ &\leq |\beta|^{-\exp(c)-\mathrm{Re}(s)+\varepsilon} && \text{(for all } \varepsilon > 0) \\ &= |\alpha|^{\exp(c)+\mathrm{Re}(s)-\varepsilon} \end{aligned}$$

Choosing some $0 < \varepsilon < \exp(c)$, we see that the integral converges on the small open vertical strip $|\mathrm{Re}(s)| < \exp(c) - \varepsilon$. Note the radius of convergence depends on c. □

Lemma 18. *For $0 < \exp(c) < 1$, define $\hat{c}(\alpha) := c(\alpha)^{-1}|\alpha|$. For any $f, g \in Z$, we have*

$$\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \zeta(\hat{f}, \hat{c})\zeta(g, c).$$

[CF77, Lemma 2.4.2]

Proof. We expand out the left hand side:

$$\begin{aligned} \zeta(f,c)\zeta(\hat{g},\hat{c}) &= \int_{K^\times} f(\alpha)c(\alpha) \, d^\times\alpha \int_{K^\times} \hat{g}(\alpha)c(\alpha)^{-1}|\alpha| \, d^\times\alpha \\ &= \int_{K^\times}\int_{K^\times} f(\alpha)\hat{g}(\beta)c(\alpha\beta^{-1})|\beta| \, d^\times\alpha d^\times\beta \\ &= \int_{K^\times}\int_{K^\times} f(\alpha)\hat{g}(\alpha\beta)c(\beta^{-1})|\alpha\beta| \, d^\times\alpha d^\times\beta && ((\alpha,\beta) \mapsto (\alpha,\alpha\beta)) \\ &= \int_{K^\times}\left(\int_{K^\times} f(\alpha)\hat{g}(\alpha\beta)|\alpha| \, d^\times\alpha\right) c(\beta)^{-1}|\beta| \, d^\times\beta. && \text{(Fubini)} \end{aligned}$$

We will be done if we can show that the inner integral is stable under swapping f and g. To do this, we begin with an integral which is obviously symmetric under switching f and g:

$$\int_{K^+}\int_{K^+} f(\xi)g(\eta)\overline{\lambda(\eta\beta\xi)} \, d\xi d\eta$$

where $\lambda \in \hat{K^+}$ is a fixed non-trivial character inducing an isomorphism $K^+ \to \hat{K^+}$.

$$\begin{aligned} \int_{K^+}\int_{K^+} f(\xi)g(\eta)\overline{\lambda(\eta\beta\xi)} \, d\xi d\eta &= \int_{K^+} f(\xi)\left(\int_{K^+} g(\eta)\overline{\lambda(\eta\beta\xi)} \, d\eta\right) d\xi \\ &= \int_{K^+} f(\xi)\hat{g}(\beta\xi) \, d\xi \\ &= (\text{constant})\int_{K^\times} f(\alpha)\hat{g}(\beta\alpha)|\alpha| \, d^\times\alpha. \end{aligned}$$

□

Theorem 19. *A ζ-function has an meromorphic continuation to the complex manifold of all quasi-characters given by a functional equation of the type*

$$\zeta(f,c) = \rho(c)\zeta(\hat{f},\hat{c}).$$

The factor $\rho(c)$ is independent of f, and is a meromorphic function on all quasi-characters. [CF77, Theorem 2.4.1]

Proof. It suffices to exhibit for each equivalence class C of quasi-characters some explicit function $f_C \in Z$ such that $\rho(c) := \zeta(f,c)/\zeta(\hat{f_C},\hat{c})$ is defined (i.e. the denominator is not identically zero) for c in the strip $0 < \exp(c) < 1$ on C. The function $\rho(c)$ defined in this manner will turn out to be a familiar meromorphic function of the parameter s having meromorphic continuation to all of C. We omit the details in these notes. $\square$

4. L-Functions

We switch gears now and restrict our attention to the space S. Fix a non-trivial character $\lambda \in \hat{K^+}$ inducing an isomorphism $K^+ \to \hat{K^+}$. Fix the unique measure on K^+ such that $\hat{\hat{f}}(\xi) = f(-\xi)$. Note that this depends on λ, since the definition of $\hat{f}$ as a function on K^+ depends on λ. This is called the *self-dual Haar measure on K^+, relative to λ*.

Let K be non-archimedean, and $\mathcal{O}$ its valuation ring. Recall that $S = S(K)$ in this case is just the space of compactly supported smooth functions on K, which we will denote $C_c^\infty(K)$. Choose a uniformizer $\varpi \in K$. Let $c : K^\times \to \mathbb{C}^\times$ be a quasi-character. Let $U \subset K^\times$ be the subset $\{\alpha \in K^\times : |\alpha| = 1\}$. For $m \in \mathbb{Z}$, let I_m denote the characteristic function of $\varpi^m U$. Note that the topology on $K^\times \subset K$ is just the subspace topology.

For $f \in C_c^\infty(K)$, the function $f_m = I_m f$ lies in $C_c^\infty(K^\times) \subset C_c^\infty(K)$. Note that f_m is identically zero for $m \ll 0$, since compact sets are bounded. We may therefore set

$$z_m = z_m(f,c) = \int_{\varpi^m U} f(\alpha)c(\alpha)\ d^\times\alpha, \qquad m \in \mathbb{Z},$$

and define a formal Laurent series $\zeta(f,c,X) \in \mathbb{C}((X))$ by the formula

$$\zeta(f,c,X) = \sum_{m\in\mathbb{Z}} z_m X^m.$$

Remark 20. *Plugging in $X = q^{-s}$ recovers the original ζ-function definition as a function of s on the component of C, possibly shifted along the x-axis.*

$$\begin{aligned}\zeta(f,c,q^{-s}) &= \sum_{m\in\mathbb{Z}} \left(\int_{\varpi^m U} f(\alpha)c(\alpha)\ d^\times\alpha\right) q^{-sm} \\ &= \sum_{m\in\mathbb{Z}} \int_{\varpi^m U} f(\alpha)c(\alpha)|\alpha|^s\ d^\times\alpha \\ &= \int_{K^\times} f(\alpha)c(\alpha)|\alpha|^s\ d^\times\alpha.\end{aligned}$$

The map $C_c^\infty(K) \to \mathbb{C}((X))$ sending $f \mapsto \zeta(f,c,X)$ is $\mathbb{C}$-linear. We denote its image

$$Z(c) = \{\zeta(f,c,X) : f \in C_c^\infty(K)\}.$$

For $a \in K^\times$ and $f \in C_c^\infty(K)$, denote by af the function $x \mapsto f(a^{-1}x)$. Then

$$\zeta(af,c,X) = c(a)X^{v(a)}\zeta(f,c,X).$$

This gives $Z(c)$ the structure of a $\mathbb{C}[X, X^{-1}]$-module. The ring $\mathbb{C}[X, X^{-1}]$ is principal ideal domain, and its unit group consists of the monomials aX^b, $a \in \mathbb{C}^\times$, $b \in \mathbb{Z}$.

Proposition 21. *Let c be a quasi-character of $K^\times$. Then*

$$Z(c) = P_c(X)^{-1}\mathbb{C}[X, X^{-1}]$$

where

$$P_c(X) = \begin{cases} 1 - c(\varpi)X & \text{if } c \text{ is unramified,} \\ 1 & \text{otherwise.} \end{cases}$$

[BH06, p.140]

Proof. Recall the inclusion $C_c^\infty(K^\times) \subset C_c^\infty(K)$. Note that the restriction $f|_{K^\times}$ of a function $f \in C_c^\infty(K)$ lies in $C_c^\infty(K^\times)$ if and only if $f(0) = 0$. In this case, $\zeta(f, c, X)$ has only finitely many non-zero coefficients. This is because if f vanishes at 0, it does so locally. So

$$\{\zeta(f, c, X) : f \in C_c^\infty(K^\times)\} \subset \mathbb{C}[X, X^{-1}].$$

In fact, this is a $\mathbb{C}[X, X^{-1}]$-submodule, i.e. an ideal. Let f be the characteristic function of a small enough neighbourhood of 1, then $\zeta(f, c, X)$ is a positive constant. So

$$\{\zeta(f, c, X) : f \in C_c^\infty(K^\times)\} = \mathbb{C}[X, X^{-1}].$$

Let f_0 be the characteristic function of $\mathcal{O}$. Then

$$\begin{aligned} \zeta(f_0, c, X) &= \sum_{m \geq 0} \left(\int_{\varpi^m U} c(\alpha)\ d^\times \alpha \right) X^m \\ &= \sum_{m \geq 0} c(\varpi^m) X^m \int_U c(\alpha)\ d^\times \alpha. \end{aligned}$$

The inner integral is $\mathrm{vol}^\times(U)$ if c is unramified, and is zero otherwise. So

$$\mathrm{vol}^\times(U)^{-1}\zeta(f_0, c, X) = \begin{cases} (1 - c(\varpi)X)^{-1} & \text{if } c \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $C_c^\infty(K)$ is the $\mathbb{C}$-span of f_0 and $C_c^\infty(K^\times)$, and since $Z(c)$ is a $\mathbb{C}[X, X^{-1}]$-module,

$$Z(c) = \mathbb{C}P_c(X)^{-1} + \mathbb{C}[X, X^{-1}] = \mathbb{C}[X, X^{-1}]P_c(X)^{-1}. \qquad \square$$

Let $\check{c} = c^{-1}$ denote the contragredient representation, i.e. smooth dual, of c. There is an analogous functional equation in this setup.

Theorem 22. *Let c be a quasi-character of $K^\times$. There exists a unique rational function $\rho(c, X) \in \mathbb{C}(X)$ such that*

$$\zeta(\hat{f}, \check{c}, \tfrac{1}{qX}) = \rho(c, X)\zeta(f, c, X)$$

for all $f \in C_c^\infty(K)$. The factor $\rho(c, X)$ depends on the choice of $0 \neq \lambda \in \hat{K^+}$ inducing $K^+ \cong \hat{K^+}$, but we have suppressed this dependence, since λ was fixed at the beginning. [BH06, p.140]

We describe how this relates to a more traditional notation. For a complex variable s:

$$\begin{aligned} \zeta(f, c, s) &:= \zeta(f, c, q^{-s}) \\ L(c, s) &:= P_c(q^{-s})^{-1} \\ \rho(c, s) &:= \rho(c, q^{-s}) \end{aligned}$$

Define a rational function

$$\varepsilon(c,s) := \rho(c,s)\frac{L(c,s)}{L(\check{c},1-s)}.$$

Proposition 23. *The function $\varepsilon(c,s)$ satisfies the functional equation*

$$\varepsilon(c,s)\varepsilon(\check{c},1-s) = c(-1).$$

For some $n(c) \in \mathbb{Z}$, one has that

$$\varepsilon(c,s) = q^{(\frac{1}{2}-s)n(c)}\varepsilon(c,\tfrac{1}{2}).$$

[BH06, p.142]

Proof. By the functional equation

$$\zeta(\hat{f},\check{c},1-s) = \rho(c,s)\zeta(f,c,s).$$

Applying this twice, we get

$$\zeta(\hat{\hat{f}},c,s) = \rho(\check{c},1-s)\rho(c,s)\zeta(f,c,s).$$

The Fourier inversion formula tells us that

$$\zeta(\hat{\hat{f}},c,s) = c(-1)\zeta(f,c,s).$$

We conclude that

$$\varepsilon(\check{c},1-s)\varepsilon(c,s) = \rho(\check{c},1-s)\rho(c) = c(-1).$$

We can rewrite the functional equation in the form

$$\frac{\zeta(\hat{f},\check{c},1-s)}{L(\check{c},1-s)} = \varepsilon(c,s)\frac{\zeta(f,c,s)}{L(c,s)}.$$

The fractions on either side lie in $\mathbb{C}[q^s,q^{-s}]$. We may choose f so that $\zeta(f,c,s) = L(c,s)$ and deduce that $\varepsilon(c,s) \in \mathbb{C}[q^s,q^{-s}]$. The functional equation implies that $\varepsilon(c,s)$ is an invertible element of $\mathbb{C}[q^s,q^{-s}]$ and hence equal to aq^{ms} for some $a \in \mathbb{C}^\times$ and $m \in \mathbb{Z}$. □

Example 24. *Let $c_0 : K^\times \to \mathbb{C}^\times$ be the unramified quasi-character $\alpha \mapsto |\alpha|^{-1}$. This corresponds to the local cyclotomic character $W_K \to \overline{\mathbb{Q}}_l^\times \cong \mathbb{C}^\times$ under the local Langlands correspondence. We compute its L-function. Let f_0 be the characteristic function of $\mathcal{O}$.*

(1) First, we use Tate's original definition.

$$\begin{aligned}\zeta(f_0,c_0|-|^s) &= \int_{K^\times} f_0(\alpha)c_0(\alpha)|\alpha|^s \, d^\times\alpha \\ &= \sum_{m\geq 0}\int_{\varpi^m U}|\alpha|^{s-1} \, d^\times\alpha \\ &= \sum_{m\geq 0} q^{(1-s)m}\int_U |\alpha|^{s-1} \, d^\times\alpha \\ &= \frac{1}{1-qq^{-s}}.\end{aligned}$$

(2) Next, we use the Laurent series definition.

$$\begin{aligned}\zeta(f_0, c_0, s) &= \int_{K^\times} f_0(\alpha)c_0(\alpha)|\alpha|^s \ d^\times\alpha \\ &= \int_{K^\times} f_0(\alpha)|\alpha|^{s-1} \ d^\times\alpha \\ &= \frac{1}{1-qq^{-s}}.\end{aligned}$$

They give the same L-function, as we expected.

References

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