SUPERCUSPIDAL REPRESENTATIONS AND THE LOCAL LANGLANDS CORRESPONDENCE

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1. Supercuspidal Representations

Let G be a locally profinite group. Let (π, V) be a smooth representation of G.

Definition 1. *The* smooth dual *of V is a representation $(\pi^\vee, V^\vee)$ of G where*

$$V^\vee := \bigcup_{\substack{K \subset G \\ \text{open compact}}} (V^*)^K \subseteq V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

Proposition 2. *If V is smooth and admissible, then*

(i) $V^\vee$ is smooth and admissible,
(ii) the canonical map $V \to (V^\vee)^\vee$ is an isomorphism,
(iii) if V is irreducible, then so is $V^\vee$.

For $v \in V$ and $\lambda \in V^\vee$, we can form their *matrix coefficient*

$$\begin{aligned} m_{v,\lambda} : G &\to \mathbb{C} \\ g &\mapsto \lambda(gv). \end{aligned}$$

Definition 3. *A smooth admissible representation (π, V) of G is called* supercuspidal *if all of its matrix coefficients are compactly supported modulo the centre, i.e. there exists a compact subset $\Omega \subset G$ such that* $\mathrm{supp}(m_{v,\lambda}) \subset Z\Omega$.

Proposition 4. *If (π, V) is irreducible, then it suffices to check that a single matrix coefficient has compact support modulo the centre.*

Proof. Since $V^\vee$ is also irreducible, any $v' \in V$, resp. $\lambda' \in V^\vee$, is a linear combination of elements of the form gv, resp. $h\lambda$, for $g, h \in G$. Then $m_{v',\lambda'}$ is a linear combination of matrix coefficients of the form

$$m_{gv,h\lambda} : x \mapsto \lambda(h^{-1}xgv)$$

which has compact support modulo the centre. □

Let $\mathbb{G}$ be a connected reductive algebraic group over a non-archimedean local field F. Consider its F-points $G = \mathbb{G}(F)$.

Proposition 5. *Let H be an open subgroup of G containing the centre, and compact modulo the centre. Let (σ, W) be an irreducible finite dimensional representation of H. If*

$$\mathrm{cInd}_H^G W := \left\{ f : G \to W \;\middle|\; \begin{array}{l} f \text{ has compact support modulo the centre} \\ \text{and } f(hg) = \sigma(h) f(g) \text{ for all } h \in H, g \in G \end{array} \right\}$$

is irreducible and admissible, then it is supercuspidal.

Proof. By irreducibility, it suffices to construct a single matrix coefficient that is compact modulo the centre. By finite-dimensionality of W, choose $0 \neq w \in W$ and $0 \neq \lambda \in W^*$ such that $\lambda(w) \neq 0$. Define $f_w \in \mathrm{cInd}_H^G W$ and $f_\lambda \in \mathrm{cInd}_H^G(W^*)$ by the formulas

$$f_w(g) = \begin{cases} \sigma(g)w & \text{if } g \in H, \\ 0 & \text{otherwise,} \end{cases} \qquad f_\lambda(g) = \begin{cases} \sigma^*(g)\lambda & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

We can view f_λ as an element of $(\mathrm{cInd}_H^G W)^\vee$ as follows: for $f \in \mathrm{cInd}_H^G W$, set

$$\langle f_\lambda, f \rangle = \langle f_\lambda(1), f(1) \rangle \in \mathbb{C}$$

where the second pairing is the canonical one between W^* and W. This identifies f_λ with the element $\langle f_\lambda, - \rangle \in (\mathrm{cInd}_H^G W)^\vee$. We now form the matrix coefficient

$$m_{f_w, f_\lambda}(g) := \langle f_\lambda, g f_w \rangle = \langle f_\lambda(1), (g f_w)(1) \rangle = \langle \lambda, f_w(g) \rangle .$$

It is non-zero, since $m_{f_w,f_\lambda}(1) = \langle \lambda, w \rangle \neq 0$. It is compactly supported modulo the centre because $\mathrm{supp}(m_{f_w,f_\lambda}) \subset \mathrm{supp}\, f_w \subset H$. □

Conjecture 6. *All supercuspidals arise in this way.*

Let $P = MN$ be the Levi decomposition of a proper parabolic subgroup P of G. Let (π, V) be a smooth admissible representation of G. Set

$$V(N) := \mathrm{span}\{\pi(n)v - v : n \in N\},$$
$$V_N := V/V(N).$$

Then M acts on V_N by $\pi|_M$.

Definition 7. *The module $J_P(V) = V_N$ with M-action given by*

$$\pi_N := \pi|_M \otimes \delta_P^{-1/2}$$

is called the Jacquet module *of (π, V) with respect to P. This is an (exact) functor*

$$J_P : \{\text{smooth } G\text{-representations}\} \to \{\text{smooth } M\text{-representations}\}.$$

Proposition 8. *J_P is left adjoint to nInd_P^G, i.e. there is an isomorphism*

$$\mathrm{Hom}_G(V, \mathrm{nInd}_P^G W) \to \mathrm{Hom}_M(J_P(V), W)$$

for all G-representations V and M-representations W.

Theorem 9 (Jacquet)**.** *(i) $J_P(V)$ is admissible if V is admissible.*
(ii) A smooth irreducible admissible representation (π, V) is supercuspidal if and only if $J_P(V) = 0$ for all proper parabolic subgroups $P \subsetneq G$.

Theorem 10. *If (π, V) is a smooth irreducible admissible representation of G, then there exists a parabolic subgroup $P \subset G$ with Levi decomposition $P = MN$ and a supercuspidal representation (σ, W) of M such that (π, V) is isomorphic to a subrepresentation of*

$$\mathrm{nInd}_P^G W.$$

Proof. Since V is irreducible, it suffices to show there exists a non-zero G-equivariant map

$$V \to \mathrm{nInd}_P^G W$$

for some (σ, W) as in the statement of the theorem. We induct on $\dim G$: the dimension of G as an algebraic group. If $\dim G = 1$, then it is a torus and equals its centre, so any function on G is compactly supported modulo the centre.

Assume $\dim G > 1$. First, assume there are no G-equivariant maps

$$V \to \mathrm{nInd}_P^G W$$

for any proper parabolic $P = MN$ and smooth admissible representation (σ, W) of M. Then by the adjunction of J_P and nInd_P^G and the fact that $J_P(V)$ is admissible, we have that $J_P(V) = 0$ for all proper parabolic subgroups P. In this case, V is supercuspidal.

Now assume there is a proper parabolic $P = MN$, a smooth admissible (not necessarily supercuspidal) representation (σ, W) of M, and a non-zero G-equivariant map

$$V \to \mathrm{nInd}_P^G W.$$

By adjunction, there is a non-zero M-equivariant map

$$J_P(V) \to W.$$

Since P is proper, we have $\dim M < \dim G$, and so our induction hypothesis implies there exists a parabolic subgroup Q of M with Levi subgroup L, a supercuspidal representation (ρ, U) of L, and a non-zero M-equivariant map

$$W \to \mathrm{nInd}_Q^M U.$$

Composing with the map $J_P(V) \to W$, and applying adjunction again, we get

$$V \to \mathrm{nInd}_P^G(\mathrm{nInd}_Q^M U).$$

It can be shown that QN is a parabolic subgroup of G with Levi subgroup L. Finally, we apply the transitivity of induction to obtain

$$\mathrm{nInd}_P^G(\mathrm{nInd}_Q^M U) = \mathrm{nInd}_{QN}^G U. \qquad \square$$

The two pictures that we are trying to paint are **(1)** "supercuspidal representations are precisely the ones that do not come from parabolic induction", i.e. they are new for G, and **(2)** "supercuspidal representations generate all irreducible admissible representations". The following definition/theorem elaborates on this idea for $G = \mathrm{GL}_n(F)$.

Theorem 11 ([GH11] 14.5.6)**.** *Let (π, V) be an irreducible smooth representation of $\mathrm{GL}_n(F)$. Then there exists a unique unordered partition $\kappa = (\kappa_1, \ldots, \kappa_r)$ of n and an unordered tuple $(\pi_1, \ldots, \pi_r)$ of supercuspidal representations, unique up to isomorphism, satisfying*

(i) π_i is a supercuspidal representation of $\mathrm{GL}_{\kappa_i}(F)$ for all $1 \le i \le r$,
(ii) π is isomorphic to a subquotient of $\mathrm{nInd}_P^G(\pi_1 \otimes \cdots \otimes \pi_r)$ where P is the standard parabolic subgroup of G associated to the partition κ.

The unordered tuple $(\pi_1, \ldots, \pi_r)$ is called the supercuspidal support *of π.*

For the rest of these notes, let $G = \mathrm{GL}_n(F)$.

Definition 12 (Segments)**.** *(i) For any representation π of $\mathrm{GL}_n(F)$, and any integer s, we write $\pi(s) := \pi \otimes |\det|^s$.*
(ii) A segment *is a set of isomorphism classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ of the form $\Delta = \{\pi, \pi(1), \ldots, \pi(r-1)\}$ for some $r \ge 1$, and we write $\Delta = [\pi, \pi(r-1)]$.*
(iii) We say that two segments Δ_1, Δ_2 are linked *if neither contains the other, and $\Delta_1 \cup \Delta_2$ is also a segment.*
(iv) If $\Delta_1 = [\pi, \pi']$ and $\Delta_2 = [\pi'', \pi''']$ are two segments, we say that Δ_1 precedes *Δ_2 if they are linked and $\pi'' = \pi(r)$ for some $r \ge 0$.*

Theorem 13 ([CEG$^+$16] Bernstein-Zelevinsky)**.** *Let $P = MN$ be the Levi decomposition of the parabolic subgroup of G associated to the partition $n = n_1 + \cdots + n_k$.*

(i) Consider $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ where each σ_i is an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$. The induction $\mathrm{nInd}_P^G \sigma$ is reducible if and only if there exists $i \neq j$ such that $n_i = n_j$ and $\sigma_i = \sigma_j(1)$.

(ii) Suppose $m = n_1 = \cdots = n_k$ so that $n = km$. The induction $\mathrm{nInd}_P^G \Delta$ of a segment $\Delta = [\pi, \pi(k-1)]$ has a unique irreducible quotient, denoted $Q(\Delta)$.

(iii) Consider segments $\{\Delta_i\}_{i=1}^k$ where each $Q(\Delta_i)$ is a representation of $\mathrm{GL}_{n_i}(F)$ and so that Δ_i does not precede Δ_j whenever $i < j$. Then the induced representation $\mathrm{nInd}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_k))$ has a unique irreducible quotient, called the Langlands quotient, *denoted either $Q(\Delta_1, \dots, \Delta_k)$ or $Q(\Delta_1) \boxplus \cdots \boxplus Q(\Delta_k)$. Any irreducible representation π of G is obtained uniquely in this way, up to a permutation of the Δ_i.*

Example 14. *Let $G = \mathrm{GL}_2(F)$. Let $P = MN$ be the Levi decomposition of the upper triangular Borel subgroup of G. We describe the Langlands quotient for each of the four families arising in the usual classification of irreducible admissible representation of $\mathrm{GL}_2(F)$. This classification can be found in* [Bum97].

(i) Let χ_1, χ_2 be characters of $F^\times$ so that $\chi_1\chi_2^{-1} \neq |-|^{\pm 1}$. Recall that in this case we have $\mathrm{nInd}_P(\chi_1 \otimes \chi_2)$ is irreducible. Let $\Delta_1 = \{\chi_1\}$ and $\Delta_2 = \{\chi_2\}$. We can write

$$\mathrm{nInd}_P(\chi_1 \otimes \chi_2) = \mathrm{nInd}_P^G(Q(\Delta_1) \otimes Q(\Delta_2)) = Q(\Delta_1) \boxplus Q(\Delta_2).$$

We remark that $Q(\Delta_1) \boxplus Q(\Delta_2) \cong Q(\Delta_2) \boxplus Q(\Delta_1)$.

(ii) Let χ_1, χ_2 be characters of $F^\times$ so that $\chi_1\chi_2^{-1} = |-|$. Then there exists a character χ such that $\chi_1 = \chi|-|^{1/2}$ and $\chi_2 = \chi|-|^{-1/2}$. There is a short exact sequence

$$0 \longrightarrow \chi \boxtimes \mathrm{St}_G \longrightarrow \mathrm{nInd}_P^G(\chi|-|^{1/2} \otimes \chi|-|^{-1/2}) \longrightarrow \chi \circ \det \longrightarrow 0.$$

Let $\Delta_1 = \{\chi|-|^{1/2}\}$ and $\Delta_2 = \{\chi|-|^{-1/2}\}$. Then

$$\chi \circ \det = Q(\Delta_1) \boxplus Q(\Delta_2).$$

We note that this is permitted, since Δ_1 does not precede Δ_2.

(iii) Let χ_1, χ_2 be characters of $F^\times$ so that $\chi_1\chi_2^{-1} = |-|^{-1}$. Then there exists a character χ such that $\chi_1 = \chi|-|^{-1/2}$ and $\chi_2 = \chi|-|^{1/2}$. There is a short exact sequence

$$0 \longrightarrow \chi \circ \det \longrightarrow \mathrm{nInd}_P^G(\chi|-|^{-1/2} \otimes \chi|-|^{1/2}) \longrightarrow \chi \boxtimes \mathrm{St}_G \longrightarrow 0\ .$$

Let $\Delta = \{\chi|-|^{-1/2}, \chi|-|^{1/2}\}$. Then

$$\chi \boxtimes \mathrm{St}_G = Q(\Delta).$$

(iv) Let σ be an irreducible supercuspidal representation of G. Let $\Delta = \{\sigma\}$. Then

$$\sigma = Q(\Delta).$$

2. Local Langlands Correspondence

Let $V = \mathbb{C}^n$. Let $N \in M_n(\mathbb{C})$ be the standard Jordan block of rank $n-1$. Let $\{v_0, \dots, v_{n-1}\}$ be the standard basis of V. Define a smooth representation ρ of W_F by $\rho(x)v_i = |x|^i v_i$ for $0 \leq i \leq n-1$ and $x \in W_F$. We form the triple (ρ, V, N) which is a semisimple Weil-Deligne representation of W_F, denoted $\mathrm{Sp}(n)$.

Let $\mathcal{G}_n(F)$ denote the set of equivalence classes of n-dimensional, semisimple, complex Weil-Deligne representations of the Weil group W_F. Let $\mathcal{A}_n(F)$ denote the set of equivalence classes of irreducible smooth representations of $G = \mathrm{GL}_n(F)$.

Theorem 15 ([BH06], [CEG+16] Local Langlands Correspondence for GL_n)**.** *Let ψ be a non-trivial additive character of F. There is a unique map*

$$\mathrm{rec} : \mathcal{A}_n(F) \to \mathcal{G}_n(F)$$

such that for all $\pi \in \mathcal{A}_n(F)$ and all characters χ of $F^\times$,

(i) $L(\chi \otimes \mathrm{rec}(\pi), s) = L(\chi \boxtimes \pi, s)$,
(ii) $\varepsilon(\chi \otimes \mathrm{rec}(\pi), s, \psi) = \varepsilon(\chi \boxtimes \pi, s, \psi)$.

The map is an isomorphism, and it respects parabolic induction in the following sense:

(i) If $\Delta = [\pi, \pi(r-1)]$ is a segment, then $\mathrm{rec}(Q(\Delta)) = \mathrm{rec}(\pi) \otimes \mathrm{Sp}(r)$,
(ii) $\mathrm{rec}(Q(\Delta_1) \boxplus \cdots \boxplus Q(\Delta_k)) = \mathrm{rec}(Q(\Delta_1)) \oplus \cdots \oplus \mathrm{rec}(Q(\Delta_k))$.

Recall that $\oplus$ and $\otimes$ are defined for Weil-Deligne representations as follows:

$$(\rho, V, N) \oplus (\sigma, W, M) = (\rho \oplus \sigma, V \oplus W, N \oplus M),$$
$$(\rho, V, N) \otimes (\sigma, W, M) = (\rho \otimes \sigma, V \otimes W, N \otimes 1 + 1 \otimes M).$$

Example 16. *Let $G = \mathrm{GL}_2(F)$. Let $P = MN$ be the Levi decomposition of the upper triangular Borel subgroup of G. We use the same notation as Example 14.*

(i) Let χ_1, χ_2 characters of $F^\times$ so that $\chi_1\chi_2^{-1} \neq |-|^{\pm 1}$.

$$\Delta_1 = \{\chi_1\}$$
$$\Delta_2 = \{\chi_2\}$$

$$\begin{aligned}\mathrm{rec}(Q(\Delta_1) \boxplus Q(\Delta_2)) &= \mathrm{rec}(Q(\Delta_1)) \oplus \mathrm{rec}(Q(\Delta_2)) \\ &= (\chi_1 \oplus \chi_2, \mathbb{C} \oplus \mathbb{C}, 0 \oplus 0) \\ &= (\chi_1 \oplus \chi_2, \mathbb{C}^2, 0)\end{aligned}$$

(ii) Let χ_1, χ_2 be characters of $F^\times$ so that $\chi_1\chi_2^{-1} = |-|$.

$$\Delta_1 = \{\chi|-|^{1/2}\}$$
$$\Delta_2 = \{\chi|-|^{-1/2}\}$$

$$\begin{aligned}\mathrm{rec}(Q(\Delta_1) \boxplus Q(\Delta_2)) &= \mathrm{rec}(Q(\Delta_1)) \oplus \mathrm{rec}(Q(\Delta_2)) \\ &= (\chi|-|^{1/2} \oplus \chi|-|^{-1/2}, \mathbb{C} \oplus \mathbb{C}, 0 \oplus 0) \\ &= (\chi|-|^{1/2} \oplus \chi|-|^{-1/2}, \mathbb{C}^2, 0)\end{aligned}$$

(iii) Let χ_1, χ_2 be characters of $F^\times$ so that $\chi_1\chi_2^{-1} = |-|^{-1}$.

$$\Delta = \{\chi|-|^{-1/2}, \chi|-|^{1/2}\}$$

$$\begin{aligned}\mathrm{rec}(Q(\Delta)) &= \mathrm{rec}(\chi|-|^{-1/2}) \otimes \mathrm{Sp}(2) \\ &= (\chi|-|^{-1/2} \otimes (1 \oplus |-|), \mathbb{C} \otimes \mathbb{C}^2, 0 \otimes 1 + 1 \otimes (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})) \\ &= (\chi|-|^{-1/2} \oplus \chi|-|^{1/2}, \mathbb{C}^2, (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}))\end{aligned}$$

(iv) Let σ be an irreducible supercuspidal representation of G. Then $\Delta = \{\sigma\}$ and

$$\mathrm{rec}(\sigma) = \mathrm{rec}(Q(\Delta)) = \mathrm{rec}(\sigma) \otimes \mathrm{Sp}(1) = \mathrm{rec}(\sigma).$$

So computing $\mathrm{rec}(\sigma)$ cannot be reduced to the case of GL_1, i.e. local class field theory. This tells us that there is some genuine work that needs to be done to figure out $\mathrm{rec}(\sigma)$. In fact, the reciprocity map rec is completely determined by the images of irreducible supercuspidal representations. See [BH06] for more discussion.

References

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