COMPLETED COHOMOLOGY AND EIGENVARIETIES

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ABSTRACT. An eigenvariety is a rigid analytic space whose points p-adically interpolate automorphic eigenforms of finite-slope. The goal of these notes is to not only give a detailed account of Emerton's particular construction of the eigenvariety via completed cohomology and the locally analytic Jacquet functor, but to also dissect and isolate the key ingredients involved in his construction, so that a first time reader has an easier time making the connection between big picture and minutiae. To this end, the fully worked out examples are, in my opinion, the most insightful parts of these notes. Chapter 1 summarizes some basic facts about finding classical automorphic forms in cohomology, and hence why cohomology is a natural place to look for a construction of the eigenvariety. Chapter 2 summarizes the theory of "algebraic modular forms", which is a space of functions defined over $\mathbb{Q}$, and so we are allowed to tensor with either $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$. In the former case, we recover a space of classical automorphic forms; in the latter case, we get something new: a space of functions which we will soon call p-adic automorphic forms. This is the first step in translating between the classical and p-adic theories of automorphic forms, and we have dubbed this dictionary the "∞-to-p switch". Chapter 3 defines completed cohomology as a p-adic completion of these algebraic modular forms, and completed cohomology is precisely the space of p-adic automorphic forms that we alluded to earlier. The space of classical automorphic forms sits inside completed cohomology as the subspace of locally algebraic vectors. Chapter 4 explains why applying the locally analytic Jacquet functor is the same as passing to the finite-slope part, and hence why eigenvarieties only parameterize finite-slope eigenforms. Finally, Chapter 5 gives Emerton's construction of the eigenvariety.

CONTENTS

1. Automorphic forms appearing in cohomology

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. It factorizes as $\mathbb{A} = \mathbb{R} \otimes \mathbb{A}^\infty$ where $\mathbb{A}^\infty$ is the ring of finite adeles. Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$. Then $G_\infty = \mathbb{G}(\mathbb{R})$ is a real Lie group. Fix a maximal compact subgroup $K_\infty \leq G_\infty$. Let K_∞^+ be its connected component of the identity. Let $A_\infty \leq G_\infty$ be the $\mathbb{R}$-points of a maximal $\mathbb{Q}$-split torus in the centre of $\mathbb{G}$. Let A_∞^+ be its connected component of the identity.

For an open compact subgroup K^∞ of $\mathbb{G}(\mathbb{A}^\infty)$, we write

$$Y(K^\infty) := \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^+ K_\infty^+ K^\infty.$$

Let $Y_\infty := \mathbb{G}(\mathbb{R})/A_\infty^+ K_\infty^+$ and $Y_\infty^+ := \mathbb{G}(\mathbb{R})^+/A_\infty^+ K_\infty^+$ where $\mathbb{G}(\mathbb{R})^+$ is the connected component of the identity of $\mathbb{G}(\mathbb{R})$. Let $\mathbb{G}(\mathbb{Q})^+ := \mathbb{G}(\mathbb{Q}) \cap \mathbb{G}(\mathbb{R})^+$. Then

$$\begin{aligned} \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^+ K_\infty^+ K^\infty &= \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{R}) \times \mathbb{G}(\mathbb{A}^\infty)/A_\infty^+ K_\infty^+ K^\infty \\ &= \mathbb{G}(\mathbb{Q})\backslash Y_\infty \times \mathbb{G}(\mathbb{A}^\infty)/K^\infty. \end{aligned}$$

Since $\mathbb{G}(\mathbb{Q})$ is dense in $\mathbb{G}(\mathbb{R})$ [Mil04, Theorem 5.4], the natural map $\mathbb{G}(\mathbb{Q})^+\backslash\mathbb{G}(\mathbb{R})^+ \hookrightarrow \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{R})$ is a bijection. Therefore, the following natural map is a bijection:

$$\mathbb{G}(\mathbb{Q})^+\backslash Y_\infty^+ \times \mathbb{G}(\mathbb{A}^\infty)/K^\infty \hookrightarrow \mathbb{G}(\mathbb{Q})\backslash Y_\infty \times \mathbb{G}(\mathbb{A}^\infty)/K^\infty.$$

Moreover, since $\mathbb{G}(\mathbb{Q})^+\backslash\mathbb{G}(\mathbb{A}^\infty)/K^\infty$ is finite [Mil04, Lemma 5.12], one has for g ranging over a finite set of representatives of $\mathbb{G}(\mathbb{Q})^+\backslash\mathbb{G}(\mathbb{A}^\infty)/K^\infty$ the following decomposition:

$$\mathbb{G}(\mathbb{Q})^+\backslash Y_\infty^+ \times \mathbb{G}(\mathbb{A}^\infty)/K^\infty = \bigsqcup_g \mathbb{G}(\mathbb{Q})^+\backslash Y_\infty^+ \times \mathbb{G}(\mathbb{Q})^+ g K^\infty/K^\infty.$$

Finally, for $g \in \mathbb{G}(\mathbb{A}^\infty)$ and $\Gamma_g^+ := gK^\infty g^{-1} \cap \mathbb{G}(\mathbb{Q})^+$, there is a natural isomorphism

$$\begin{aligned} \Gamma_g^+\backslash Y_\infty^+ &\to \mathbb{G}(\mathbb{Q})^+\backslash Y_\infty^+ \times \mathbb{G}(\mathbb{Q})^+ g K^\infty/K^\infty \\ [y] &\mapsto [y, g]. \end{aligned}$$

We have exhibited $Y(K^\infty)$ as a finite union of its connected components:

$$Y(K^\infty) = \bigsqcup_g \Gamma_g^+\backslash Y_\infty^+.$$

This space $Y(K^\infty)$ is an example of a *locally symmetric space*.

Remark 1.1. If $\mathbb{G}$ is a reductive group over a number field F, we can consider its Weil restriction $\mathbb{G}' = \operatorname{Res}_{\mathbb{Q}}^F \mathbb{G}$, and apply the above construction to the algebraic group $\mathbb{G}'$ over $\mathbb{Q}$.

Example 1.2. Let $\mathbb{G} = \operatorname{GL}_2/\mathbb{Q}$. Then $K_\infty^+ = \operatorname{SO}_2(\mathbb{R})$ and $A_\infty^+ = \mathbb{R}_{>0}$. Since $\mathbb{R}_{>0}\operatorname{SO}_2(\mathbb{R}) = \operatorname{GL}_2(\mathbb{R})_i$ is the stabilizer of i for the action $\operatorname{GL}_2(\mathbb{R}) \curvearrowright \mathbb{C}$, the homogeneous space $Y_\infty = \operatorname{GL}_2(\mathbb{R})/\operatorname{GL}_2(\mathbb{R})_i$ is equal to the orbit $\operatorname{GL}_2(\mathbb{R})i = \mathbb{H}^\pm$. Thus $Y_\infty^+ = \mathbb{H}$.

Let K^∞ be either $K_0(N)$ or $K_1(N)$ defined as follows:

$$K_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2(\widehat{\mathbb{Z}}) \;\middle|\; c \equiv 0 \pmod{N} \right\},$$

$$K_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2(\widehat{\mathbb{Z}}) \;\middle|\; \begin{matrix} c \equiv 0 \pmod{N} \\ d \equiv 1 \pmod{N} \end{matrix} \right\}.$$

I claim that the space $\mathrm{GL}_2(\mathbb{Q})^+\backslash \mathrm{GL}_2(\mathbb{A}^\infty)/K^\infty$ is a singleton. To see this, note that the determinant map $\det : \mathrm{GL}_2(\mathbb{A}^\infty) \to \mathbb{A}^{\infty,\times}$ induces a surjection:

$$\mathrm{GL}_2(\mathbb{Q})^+\backslash \mathrm{GL}_2(\mathbb{A}^\infty)/K^\infty \xrightarrow{\det} \mathbb{Q}_{>0}\backslash\mathbb{A}^{\infty,\times}/\det(K^\infty) = \widehat{\mathbb{Z}}^\times/\det(K^\infty).$$

However, $\det(K^\infty) = \widehat{\mathbb{Z}}^\times$ so the codomain is a singleton. It suffices to show that the preimage of this singleton is a singleton, that is, the following double quotient is a singleton:

$$\mathrm{GL}_2(\mathbb{Q})^+\backslash \mathrm{GL}_2(\mathbb{Q})^+\det{}^{-1}(1)K^\infty/K^\infty.$$

Indeed, $\det^{-1}(1) = \mathrm{SL}_2(\mathbb{A}^\infty)$ and so

$$\begin{aligned}\mathrm{GL}_2(\mathbb{Q})^+\backslash \mathrm{GL}_2(\mathbb{Q})^+ \mathrm{SL}_2(\mathbb{A}^\infty)K^\infty/K^\infty &= \mathrm{GL}_2(\mathbb{Q})^+ \cap \mathrm{SL}_2(\mathbb{A}^\infty)\backslash \mathrm{SL}_2(\mathbb{A}^\infty)K^\infty/K^\infty \\ &= \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}^\infty)K^\infty/K^\infty \\ &= \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}^\infty)/\mathrm{SL}_2(\mathbb{A}^\infty)\cap K^\infty.\end{aligned}$$

However, the embedding $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathrm{SL}_2(\mathbb{A}^\infty)$ is dense by the strong approximation theorem [Mil04, Theorem 4.16] and $\mathrm{SL}_2(\mathbb{A}^\infty)\cap K^\infty \subset \mathrm{SL}_2(\mathbb{A}^\infty)$ is open. Therefore,

$$\mathrm{SL}_2(\mathbb{A}^\infty) = \mathrm{SL}_2(\mathbb{Q})\left[\mathrm{SL}_2(\mathbb{A}^\infty)\cap K^\infty\right].$$

This implies that the double quotient is a singleton. Finally, it is straightforward to see that

$$\begin{aligned}\Gamma_0(N) &= K_0(N) \cap \mathrm{GL}_2(\mathbb{Q})^+ \subset \mathrm{SL}_2(\mathbb{Z}),\\ \Gamma_1(N) &= K_1(N) \cap \mathrm{GL}_2(\mathbb{Q})^+ \subset \mathrm{SL}_2(\mathbb{Z}).\end{aligned}$$

Therefore, $Y(K^\infty)$ has one connected component, equal to the modular curve
(i) $\Gamma_0(N)\backslash\mathbb{H}$ if $K^\infty = K_0(N)$,
(ii) $\Gamma_1(N)\backslash\mathbb{H}$ if $K^\infty = K_1(N)$.

We want to realize automorphic forms in the cohomology of these locally symmetric spaces. That is, we want to realize automorphic forms as classes in either singular cohomology

$$H^n(Y(K^\infty),\mathbb{C})$$

or more generally, in the cohomology of a local system $\mathcal{F}$ on $Y(K^\infty)$:

$$H^n(Y(K^\infty),\mathcal{F}).$$

Let V be a finite-dimensional complex vector space equipped with a representation of the algebraic group $\mathbb{G}/\mathbb{C}$. For an open compact $K^\infty \le \mathbb{G}(\mathbb{A}^\infty)$, we define the space

$$\mathbb{G}(\mathbb{Q})\backslash V \times Y_\infty \times \mathbb{G}(\mathbb{A}^\infty)/K^\infty.$$

The action of $\mathbb{G}(\mathbb{Q})$ is on all three factors, and K^∞ on the rightmost factor. There is a natural map from this space to $Y(K^\infty)$ induced by projecting to the right two factors:

$$\mathbb{G}(\mathbb{Q})\backslash V \times Y_\infty \times \mathbb{G}(\mathbb{A}^\infty)/K^\infty \to \mathbb{G}(\mathbb{Q})\backslash Y_\infty \times \mathbb{G}(\mathbb{A}^\infty)/K^\infty.$$

Let $\mathcal{F}_V$ be the sheaf of sections associated to this map. It is a local system on $Y(K^\infty)$.

Theorem 1.3. *Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$ such that $\mathbb{G}^{\mathrm{der}}$ is $\mathbb{Q}$-anisotropic. Let V be a finite-dimensional complex algebraic representation of $\mathbb{G}(\mathbb{C})$. Then for all $n \ge 0$, there is a decomposition of modules for the non-archimedean Hecke algebra as follows:*

$$H^n(Y(K^\infty),\mathcal{F}_V) \cong \bigoplus_\pi (\pi^\infty)^{K^\infty} \otimes H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+; \pi_\infty \otimes V)^{m(\pi)}$$

where $\mathfrak{a}_\infty$ is the Lie algebra of A_∞^+, π varies over automorphic representations of $\mathbb{G}$ such that the central character of $\pi_\infty^\vee$ restricted to A_∞^+ is equal to $V|_{A_\infty^+}$, and $m(\pi)$ is the automorphic multiplicity of π.
[GH23, §15.5] [You19, Theorem 1.53]

The choice of K_∞^+ in $(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+)$-cohomology comes from the definition of our locally symmetric space $Y(K^\infty)$ which involved a quotient by the subgroup K_∞^+. In general, we have that if $\mathfrak{g} = \mathrm{Lie}(G_\infty)$ and $\mathfrak{k} = \mathrm{Lie}(K_\infty)$, then $H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, \mathfrak{k}; -) = H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+; -)$. However, the space $H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty; -)$ can be smaller, if K_∞ is not connected.

Let $Y := \varprojlim_{K^\infty} Y(K^\infty)$ where the projective limit is taken over the directed set of all open compact subgroups $K^\infty \subset \mathbb{G}(\mathbb{A}^\infty)$ with suitable transition maps in the category of smooth real manifolds. Then we get the following:

Theorem 1.4. *Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$ such that $\mathbb{G}^{\mathrm{der}}$ is $\mathbb{Q}$-anisotropic. Let V be a finite-dimensional complex algebraic representation of $\mathbb{G}(\mathbb{C})$. Then for all $n \geq 0$, there is a decomposition of $\mathbb{G}(\mathbb{A}^\infty)$-representations as follows:*

$$H^n(Y, \mathcal{F}_V) \cong \bigoplus_\pi \pi^\infty \otimes H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+; \pi_\infty \otimes V)^{m(\pi)}$$

where $\mathfrak{a}_\infty$ is the Lie algebra of A_∞^+, π varies over automorphic representations of $\mathbb{G}$ such that the central character of $\pi_\infty^\vee$ restricted to A_∞^+ is equal to $V|_{A_\infty^+}$, and $m(\pi)$ is the automorphic multiplicity of π.
[GH23, §15.5] [You19, Theorem 1.53]

A formula of this type is called *Matsushima's formula*: these formulas say that we can find automorphic representations in the cohomology of locally symmetric spaces. The variant we described above only applies to groups $\mathbb{G}$ where $\mathbb{G}^{\mathrm{der}}$ is $\mathbb{Q}$-anisotropic. However, we will mostly be working with groups where this condition is satisfied.

Definition 1.5. An algebraic group $\mathbb{G}$ over F is called F-anisotropic if there does not exist an embedding $\mathbb{G}_{m,F} \hookrightarrow \mathbb{G}$ of algebraic groups.

Theorem 1.6. *Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$. Then the space $Y(K^\infty)$ is compact for some K^∞ (equivalently, for all K^∞) if and only if $\mathbb{G}^{\mathrm{der}}$ is $\mathbb{Q}$-anisotropic.*
[You19, Theorem 1.32]

Each locally symmetric space attached to $\mathbb{G}$ is the disjoint union of *symmetric spaces* attached to $\mathbb{G}$. There are two natural compactifications of these symmetric spaces called the *Baily-Borel compactification* and the *Borel-Serre compactification*; see Goresky's chapter in [AEK05]. The former only works if your symmetric space has the structure of a Hermitian symmetric domain, and the latter works in general. The boundary components of these compactifications are indexed by proper parabolic subgroups of $\mathbb{G}$. Therefore, the locally symmetric spaces for $\mathbb{G}$ are compact if and only if $\mathbb{G}$ has no proper parabolic subgroups. This is thus equivalent to $\mathbb{G}^{\mathrm{der}}$ being $\mathbb{Q}$-anisotropic by the preceding result.

Remark 1.7. If $\mathbb{G}^{\mathrm{der}}$ is $\mathbb{Q}$-anisotropic, then every automorphic form of $\mathbb{G}$ is *cuspidal*, since $\mathbb{G}$ has no proper parabolic subgroups.

This condition of being $\mathbb{Q}$-anisotropic is satisfied, for example, when G_∞ is compact, since in this case the locally symmetric space attached to $\mathbb{G}$ is just a finite collection of points.

We remark that being $\mathbb{Q}$-anisotropic is actually quite restrictive, since it does not even apply to the groups $\mathbb{G} = \mathrm{GL}_n/\mathbb{Q}$. Indeed, as we computed above, the locally symmetric spaces attached to $\mathrm{GL}_2/\mathbb{Q}$ are not compact. There is a more general version of Matsushima's formula which holds for more general groups, but in this generality we are only able to recover the cuspidal part of the automorphic spectrum in cohomology. Apparently, there is an even more general version called *Franke's theorem* which gives a complicated description of the entire automorphic spectrum in cohomology. We shall not discuss this at all, but it is briefly mentioned in [CE12]. In this next example below, we illustrate the idea of Matsushima's formula for $\mathrm{GL}_2/\mathbb{Q}$ via cuspidal cohomology, following the exposition in [You19].

Example 1.8. Let $\mathbb{G} = \mathrm{GL}_2/\mathbb{Q}$. Let $Y(N) := Y(K_1(N))$ and $j : Y(N) \hookrightarrow X(N)$ be the standard (Baily-Borel) compactification of $Y(N)$ by adding in the cusps. For $k \geq 2$, let $V_k := \mathrm{Sym}^{k-2}\mathbb{Q}^2$ be the representation of $\mathrm{GL}_2/\mathbb{Q}$ where $\mathbb{Q}^2$ is its standard representation.

Remark 1.9. For a cusp form $f \in S_k(\Gamma_1(N))$ with nebentypus χ, we let $\phi_{f,s}$ denote the automorphic form on $\mathrm{GL}_2/\mathbb{Q}$ associated to it, for some choice s of normalization for the power of the determinant taken in the formula for $\phi_{f,s}$. Refer to [Fen22, §4.9] for the formula. Let ω be the adelic character associated to χ. Then the centre $\mathbb{A}^\times$ of $\mathrm{GL}_2(\mathbb{A})$ acts via right translation on $\phi_{f,s}$ through the following central character, for $z \in \mathbb{A}^\times$:

$$z \mapsto |z|^{2s-k}\omega(z).$$

If $s = k/2$, then this is called the *unitary normalization*, since the central character is unitary. If $s = 1$, then this is called the *arithmetic normalization.* Since ω is trivial on $\mathbb{R}_{>0}$, one has that for $r \in \mathbb{R}_{>0} = A_\infty^+$ in the centre of $\mathrm{GL}_2(\mathbb{R})$, the central character restricts to:

$$r \mapsto r^{2-k}.$$

This cancels out the central character of $V_k \otimes_{\mathbb{Q}} \mathbb{R}$ when viewed as a representation of $\mathrm{GL}_2(\mathbb{R})$. For the rest of this example, we choose the arithmetic normalization, and so set $\phi_f := \phi_{f,1}$. Let π_f be the automorphic representation generated by ϕ_f in the appropriate space.

We define the cuspidal, or parabolic cohomology, of $\mathrm{GL}_2/\mathbb{Q}$ to be the space:

$$H^1_!(Y(N), \mathcal{F}_{V_k}) := H^1(X(N), j_*\mathcal{F}_{V_k}).$$

This has the following decomposition as modules for the non-archimedean Hecke algebra:

$$\begin{aligned} H^1(X(N), j_*\mathcal{F}_{V_k}) &= \bigoplus_\pi (\pi^\infty)^{K_1(N)} \otimes H^1(\mathfrak{a}_\infty\backslash\mathfrak{gl}_2, \mathfrak{so}_2; \pi_\infty \otimes V_k)^{m(\pi)} \\ &= \bigoplus_f (\pi_f^\infty)^{K_1(N)} \otimes H^1(\mathfrak{a}_\infty\backslash\mathfrak{gl}_2, \mathfrak{so}_2; \pi_\infty \otimes V_k). \end{aligned}$$

The fact that $m(\pi) = 1$ is a consequence of *multiplicity one* for GL_2. The first sum runs over cuspidal automorphic representations π of GL_2 containing a non-zero $K_1(N)$-fixed vector. In the second sum, π_f is the automorphic representation associated to f, and it runs over cuspidal newforms f of level $M \mid N$, that is, all of the newforms of some level in $S_k(\Gamma_1(N))$. To any cusp form f, let $f_d(z) := f(dz)$. If f is a newform of level $N_f \mid N$, then one has:

$$(\pi_f^\infty)^{K_1(N)} = \bigoplus_{dN_f \mid N} \mathbb{C}\phi_{f_d}.$$

When $k = 2$, we recover the *Eichler-Shimura isomorphism*, which we explain now. In this case, we have that $\mathcal{F}_{V_2} = \underline{\mathbb{C}}$ and thus $j_*\underline{\mathbb{C}} = \underline{\mathbb{C}}$ so it suffices to compute singular cohomology.

$$H^1(X(N), \mathbb{C}) = \bigoplus_f \left(\bigoplus_{dN_f | N} \mathbb{C}\phi_{f_d} \right) \otimes H^1(\mathfrak{a}_\infty \backslash \mathfrak{gl}_2, \mathfrak{so}_2; \pi_\infty).$$

The $(\mathfrak{a}_\infty \backslash \mathfrak{gl}_2, \mathfrak{so}_2)$-cohomology groups for π_∞ in the above sum are two-dimensional, and moreover, by the theory of newforms, for any $k \geq 2$, there is a basis:

$$S_k(\Gamma_1(N)) = \bigoplus_f \bigoplus_{dN_f | N} \mathbb{C} f_d.$$

Therefore, there is a decomposition of modules for the non-archimedean Hecke algebra:

$$H^1(X(N), \mathbb{C}) \cong S_2(\Gamma_1(N)) \oplus S_2(\Gamma_1(N)).$$

Alternatively, via Hodge theory, one has a canonical decomposition:

$$H^1(X(N), \mathbb{C}) = H^0(X(N), \Omega^1_{X(N)}) \oplus \overline{H^0(X(N), \Omega^1_{X(N)})}.$$

Recall the usual map $f \mapsto f\,dz$ inducing an isomorphism of Hecke modules:

$$S_2(\Gamma_1(N)) \to H^0(X(N), \Omega^1_{X(N)}).$$

This recovers the Eichler-Shimura isomorphism:

$$H^1(X(N), \mathbb{C}) \cong S_2(\Gamma_1(N)) \oplus \overline{S_2(\Gamma_1(N))}.$$

If $\pi_\infty = \mathcal{D}_k$ is a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $k \geq 2$, then when restricted to $\mathrm{SL}_2(\mathbb{R})$ it breaks up into two irreducible representations called the holomorphic and antiholomorphic discrete series $\mathcal{D}_k = \mathcal{D}_k^+ \oplus \mathcal{D}_k^-$. This induces a decomposition:

$$H^1(\mathfrak{a}_\infty \backslash \mathfrak{gl}_2, \mathfrak{so}_2; \mathcal{D}_k) = H^1(\mathfrak{a}_\infty \backslash \mathfrak{gl}_2, \mathfrak{so}_2; \mathcal{D}_k^+) \oplus H^1(\mathfrak{a}_\infty \backslash \mathfrak{gl}_2, \mathfrak{so}_2; \mathcal{D}_k^-)$$

in which both of the summands are one-dimensional. Indeed, the right-hand side makes sense because $\mathfrak{a}_\infty \backslash \mathfrak{gl}_2 \cong \mathfrak{sl}_2$ and this preserves $\mathcal{D}_k^+$ and $\mathcal{D}_k^-$. The summands corresponding to the holomorphic and antiholomorphic discrete series ($\mathcal{D}_2^+$ and $\mathcal{D}_2^-$) should match up with $S_2(\Gamma_1(N))$ and $\overline{S_2(\Gamma_1(N))}$ from the Eichler-Shimura isomorphism.

It now makes sense to ask which automorphic forms show up as classes inside cohomology, and with which local system they appear. This leads us to the following notion.

Definition 1.10. An automorphic representation π of $\mathbb{G}$ is called *cohomological* if there exists a complex algebraic representation V of $\mathbb{G}(\mathbb{C})$ such that

$$H^n(\mathfrak{a}_\infty \backslash \mathfrak{g}, K_\infty^+; \pi_\infty \otimes V) \neq 0$$

for some $n \geq 0$. In this case, π is said to be *V-cohomological*.

If π is cohomological, it is an interesting question as to which V can be chosen so that π is V-cohomological. The following result provides a necessary condition for such a V.

Theorem 1.11. *If π is V-cohomological, then $\pi^\vee$ has the same infinitesimal character as V.*
[GH23, Theorem 15.5.1]

Example 1.12. Let π be an automorphic representation of $\mathbb{G} = \mathrm{GL}_2/\mathbb{Q}$ where $\pi_\infty = \mathcal{D}_{k,\mu}$ for some $\mu \in \mathbb{C}$ is a discrete series $(\mathfrak{g}, K_\infty)$-module of weight $k \geq 2$, for $\mathfrak{g} = \mathfrak{gl}_2$ and $K_\infty = O(2)$. In this case, we show that there is a natural choice for V so that π is $V^\vee$-cohomological.

Let us recall some facts about discrete series. The centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is generated by two elements: the Casimir operator Δ and $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For $(s, \mu) \in \mathbb{C}^2$, define a character $\chi_{s,\mu}$ of $Z(\mathfrak{g})$ by sending $Z \mapsto \mu$ and $\Delta \mapsto s(1-s)$. For $\varepsilon \in \{0, 1\}$, there is a (possibly reducible) $(\mathfrak{g}, K_\infty)$-module $(\pi, V) = (\pi_{s,\mu,\varepsilon}, V_{s,\mu,\varepsilon})$ with infinitesimal character $\chi_{s,\mu}$ and K_∞-types:

$$\{l \in \mathbb{Z} : l \equiv \varepsilon \pmod 2\}.$$

It is irreducible unless $s = k/2$ and k is congruent to $\varepsilon \pmod 2$. If $k \geq 1$, then $\pi_{k/2,\mu,\varepsilon}$ has a unique infinite-dimensional irreducible subrepresentation $\mathcal{D}_{k,\mu}$ with K_∞-types:

$$\{l \in \mathbb{Z} : l \equiv \varepsilon \pmod 2, |l| \geq k\}.$$

The quotient $\pi_{k/2,\mu,\varepsilon}/\mathcal{D}_{k,\mu}$ is irreducible and finite-dimensional, with K_∞-types:

$$\{l \in \mathbb{Z} : l \equiv \varepsilon \pmod 2, |l| < k\}.$$

This has highest weight $k-2$, since $l \equiv \varepsilon \equiv k \pmod 2$. There is a short exact sequence:

$$0 \to \mathcal{D}_{k,\mu} \to \pi_{k/2,\mu,\varepsilon} \to \mathrm{Sym}^{k-2}\,\mathbb{C}^2 \otimes |\det|^{(\mu-(k-2))/2} \to 0.$$

For more details on these constructions, see [GH23, §4.7] and [Bum97, §2.5]. Let us calculate the infinitesimal character of the finite-dimensional representation, and show that it is $\chi_{k/2,\mu}$. The highest weight λ of the finite-dimensional representation is

$$\lambda = [k-2, 0] + [\tfrac{\mu-(k-2)}{2}, \tfrac{\mu-(k-2)}{2}] = [\tfrac{\mu+(k-2)}{2}, \tfrac{\mu-(k-2)}{2}].$$

We remark that λ is algebraic, i.e. $\lambda \in X^\bullet(T)$, if and only if $\mu \equiv k \pmod 2$. Let us fix a choice of positive roots Φ^+ for $\mathfrak{g} = \mathfrak{gl}_2$. The choice is not important, but let us choose Φ^+ to be the $\mathbb{Z}_{\geq 0}$-span of $\Delta = \{e_1 - e_2\}$. Let $\rho = \frac{1}{2}(e_1 - e_2)$ denote the half-sum of positive roots. There is an embedding $Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{t}) \oplus U(\mathfrak{g})\mathfrak{n}^+ \subset U(\mathfrak{g})$. Recall that the Harish-Chandra isomorphism is induced from the following composition of maps:

$$\begin{aligned} t_\rho : U(\mathfrak{t}) &\to U(\mathfrak{t}) \\ X &\mapsto X - \rho(X) \end{aligned}$$

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\text{incl.}"] \arrow[rrr, bend right, "\mathrm{HC}"'] & U(\mathfrak{t}) \oplus U(\mathfrak{g})\mathfrak{n}^+ \arrow[r, "\text{proj}"] & U(\mathfrak{t}) \arrow[r, "t_\rho"] & U(\mathfrak{t}) \end{tikzcd}$$

The action of $\omega \in Z(\mathfrak{g})$ on the highest weight vector v is:

$$\omega v = \mathrm{proj}(\omega) v = \lambda(\mathrm{proj}(\omega)) v = (\lambda + \rho)(\mathrm{HC}(\omega)) v.$$

Proposition 1.13. *Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be a homomorphism of associative algebras. Then $\chi = \chi_\lambda$ where $\chi_\lambda = \lambda \circ \mathrm{HC}$ for some character λ of $\mathfrak{t}$.*
[GH23, Proposition 4.6.2]

Therefore, $\chi = \chi_{\lambda+\rho}$ is the infinitesimal character of the finite-dimensional representation associated to the discrete series representation $\mathcal{D}_{k,\mu}$. One calculates:

$$\lambda + \rho = [\tfrac{\mu+(k-1)}{2}, \tfrac{\mu-(k-1)}{2}].$$

To complete our calculation, we shall use an explicit description of Δ.

$$\mathfrak{gl}_2 = \mathbb{C}Z \oplus \mathbb{C}H \oplus \mathbb{C}E \oplus \mathbb{C}F \qquad Z(\mathfrak{gl}_2) = \mathbb{C}[Z, \Delta]$$

$$E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \quad F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \quad Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

$$\Delta = -\tfrac{1}{4}(H^2 + 2EF + 2FE) = -\tfrac{1}{4}(H^2 + 2H + 4FE) = -\tfrac{1}{4}(H^2 - 2H + 4EF)$$

Note that $\Delta = -\frac{1}{4}(H^2 + 2H + 4FE) \in U(\mathfrak{t}) \oplus U(\mathfrak{g})\mathfrak{n}^+$ and $t_\rho(H) = H - 1$.

$$\begin{aligned} \mathrm{HC}(Z) &= Z \\ \mathrm{HC}(\Delta) &= -\tfrac{1}{4}(t_\rho(H)^2 + 2t_\rho(H)) = -\tfrac{1}{4}(H^2 - 1) \\ (\lambda + \rho)(\mathrm{HC}(Z)) &= \tfrac{\mu + (k-1)}{2} + \tfrac{\mu - (k-1)}{2} = \mu \\ (\lambda + \rho)(\mathrm{HC}(\Delta)) &= -\tfrac{1}{4}((k-1)^2 - 1) = \tfrac{k}{2}(1 - \tfrac{k}{2}) \end{aligned}$$

The key takeaway from this discussion is that the infinitesimal centre acts in the same way on all three factors of the short exact sequence, so for $\mathcal{D}_{k,\mu}$ there is a natural choice of a finite-dimensional representation V so that $\mathcal{D}_{k,\mu}$ is possibly $V^\vee$-cohomological, since they have the same infinitesimal character. Indeed, we saw previously that $\mathcal{D}_{k,\mu}$ is $V^\vee$-cohomological.

Obviously, μ must be chosen in such a way so that the finite-dimensional representation associated to $\mathcal{D}_{k,\mu}$ is algebraic as well, meaning it has an algebraic highest weight λ. We can detect this with a condition on the infinitesimal character.

Definition 1.14. An irreducible admissible representation π of $\mathbb{G}(\mathbb{R})$ having infinitesimal character χ_λ is called *C-algebraic* if it satisfies $\lambda - \rho \in X^\bullet(T)$.
[GH23, Lemma 12.8.1]

Corollary 1.15. *Let π be a cohomological cuspidal automorphic representation of $\mathbb{G}$. Then π_∞ is C-algebraic.*
[GH23, Corollary 15.5.2]

Recall that the discrete series $\mathcal{D}_{k,\mu}$ has infinitesimal character $\chi_{\lambda+\rho}$. So for $\mathcal{D}_{k,\mu}$ to be cohomological, it is necessary that $\lambda = (\lambda + \rho) - \rho$ be algebraic. But this is reasonable, since λ is the highest weight of the finite-dimensional representation associated to $\mathcal{D}_{k,\mu}$.

We can also ask the more elementary question of which automorphic representations π are cohomological at all. The answer to this is well-known if π_∞ is *tempered*. Let $\mathbb{G}$ be a reductive group over $\mathbb{R}$, then define quantities:

$$\begin{aligned} X^\bullet(\mathbb{G}) &:= \mathrm{Hom}(\mathbb{G}, \mathbb{G}_m) \\ \mathbb{G}(\mathbb{R})^1 &:= \bigcap_{\chi \in X^\bullet(\mathbb{G})} \ker(|-| \circ \chi : \mathbb{G}(\mathbb{R}) \to \mathbb{R}_{>0}). \end{aligned}$$

Definition 1.16. If π is an admissible representation of $\mathbb{G}(\mathbb{R})$ on a complex Hilbert space V equipped with inner product $\langle -, - \rangle$, then a function $\mathbb{G}(\mathbb{R}) \to \mathbb{C}$ is called a *matrix coefficient* of π if it is of the form $g \mapsto \langle \pi(g)v, w \rangle$ for some $v, w \in V$. More generally, let $\pi^\vee$ denote the contragredient representation of π, and consider the canonical bilinear pairing $B : \pi \times \pi^\vee \to \mathbb{C}$ where $B(v, w) = w(v)$. Then a function $\mathbb{G}(\mathbb{R}) \to \mathbb{C}$ is called a *matrix coefficient* of π if it is of the form $g \mapsto B(\pi(g)v, w)$ for some $v \in \pi$ and $w \in \pi^\vee$.

Definition 1.17. Let $\mathbb{G}$ be a reductive group over $\mathbb{R}$. An irreducible admissible representation π of $\mathbb{G}(\mathbb{R})$ with unitary central character is called *tempered* if all matrix coefficients lie in $L^{2+\varepsilon}(Z(\mathbb{R})\backslash\mathbb{G}(\mathbb{R}))$ for all $\varepsilon > 0$. Otherwise, π is called *essentially tempered* if all matrix coefficients of $\pi|_{\mathbb{G}(\mathbb{R})^1}$ lie in $L^{2+\varepsilon}((Z(\mathbb{R}) \cap \mathbb{G}(\mathbb{R})^1)\backslash\mathbb{G}(\mathbb{R})^1)$ for all $\varepsilon > 0$.

Note that the definition of essentially tempered implicitly assumes that $Z(\mathbb{R})\cap\mathbb{G}(\mathbb{R})^1$ acts unitarily on $\pi|_{\mathbb{G}(\mathbb{R})^1}$. This is justified by the following lemma and proposition.

Lemma 1.18. *Let Z be the centre of $\mathbb{G}$ as an algebraic group over $\mathbb{R}$. Then every character of Z has a power that extends to a character of $\mathbb{G}$.*

Proof. Since $\mathbb{G}$ is reductive, there is a central isogeny with finite kernel K:

$$Z \times \mathbb{G}^{\mathrm{der}} \twoheadrightarrow \mathbb{G}.$$

Let α be a character of Z, and extend it trivially to a character $\alpha \times 1$ of $Z \times \mathbb{G}^{\mathrm{der}}$. Since K is finite, one can choose $n \geq 1$ so that $(\alpha \times 1)^n$ is trivial on K. Then $(\alpha \times 1)^n$ factors via the first isomorphism theorem through a character β of $\mathbb{G}$ as follows:

$$\begin{tikzcd} Z \times \mathbb{G}^{\mathrm{der}} \arrow[r, two heads] \arrow[dr, "(\alpha\times 1)^n"'] & \mathbb{G} \arrow[d, "\beta"] \\ & \mathbb{G}_m. \end{tikzcd}$$

It is clear that $\beta|_Z = \alpha^n$. $\square$

Proposition 1.19. $Z(\mathbb{R}) \cap \mathbb{G}(\mathbb{R})^1 = Z(\mathbb{R})^1$.

Proof. Since Z is a reductive group over $\mathbb{R}$, it makes sense to write $Z(\mathbb{R})^1$ by our definition. Suppose $g \in Z(\mathbb{R})\cap\mathbb{G}(\mathbb{R})^1$, then $g \in Z(\mathbb{R})$ and $|\lambda(g)| = 1$ for all $\lambda \in X^\bullet(\mathbb{G})$. Let $\mu \in X^\bullet(Z)$, then by the previous lemma, there exists $n \geq 1$ so that $\mu^n = \lambda|_Z$ for some $\lambda \in X^\bullet(\mathbb{G})$. However, $|\lambda(g)| = 1$ by assumption, so $|\mu(g)^n| = 1$. This implies $|\mu(g)| = 1$. Therefore:

$$Z(\mathbb{R}) \cap \mathbb{G}(\mathbb{R})^1 \subset Z(\mathbb{R})^1.$$

The other direction is trivial. $\square$

Therefore, if α is any algebraic character of $Z(\mathbb{R})$, its restriction to $Z(\mathbb{R})\cap\mathbb{G}(\mathbb{R})^1 = Z(\mathbb{R})^1$ is automatically unitary by the definition of $Z(\mathbb{R})^1$.

Indeed, tempered representations are essentially tempered, and being essentially tempered is insensitive to twists by characters of $\mathbb{G}(\mathbb{R})$. Therefore, if π is an irreducible admissible representation of $\mathbb{G}(\mathbb{R})$ with a twist that is tempered and has unitary central character, then π is essentially tempered.

If π is an smooth irreducible admissible representation of $\mathbb{G}(F)$ where F is a characteristic zero local field, and $\mathbb{G}$ is a reductive group over F, then I believe you can always find a twist of it with unitary central character. See [Cas95, Lemma 5.2.5] for a proof of this fact when F is non-archimedean. We remark that if $\mathbb{G} = \mathrm{GL}_2/\mathbb{Q}$, then actually every cuspidal automorphic representation of $\mathbb{G}$ has a global twist with global unitary central character.

Let $d := \dim_{\mathbb{R}}(Y_\infty)$ and define the quantities:

$$l_0 := \operatorname{rk} G_\infty - \operatorname{rk} A_\infty^+ K_\infty^+$$
$$q_0 := (d - l_0)/2.$$

This notation appears in [CE12, §1.6] and [Eme14, §2.1]. When G_∞ is semisimple, so that in particular A_∞ is trivial, this notation coincides with that found in [BW00, §4.3].

Theorem 1.20. *Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$. Let π be an automorphic representation of $\mathbb{G}$ such that π_∞ is tempered. Then*

(i) $H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+; \pi_\infty \otimes V) = 0$ *for all finite-dimensional representations V and for all $n \notin [q_0, q_0 + l_0]$.*

(ii) If $H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+; \pi_\infty \otimes V) \neq 0$ *for some finite-dimensional representation V, then π_∞ is* fundamentally tempered*, i.e. it is induced from a discrete series representation of the Levi subgroup of a fundamental parabolic subgroup of G_∞.*

[BW00, Theorem 5.1], [Eme14]

I believe the exact same statements should hold for when π_∞ is essentially tempered, except maybe one has to be careful how to generalize the idea of being fundamentally tempered. When G_∞ is semisimple, [BW00, Theorem 5.1] provides a formula to calculate the dimensions of these cohomology groups in terms of n, l_0 and q_0.

Remark 1.21. There are tempered representations which **do not** show up in cohomology. For $k = 1$, the "discrete series" representations $\mathcal{D}_{1,\mu}$ introduced earlier are more commonly referred to as *limit of discrete series*. They are (essentially) tempered and correspond to weight $k = 1$ classical modular forms. They are not cohomological.

Remark 1.22. $(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+)$-cohomology satisfies Poincaré duality, that is

$$H^n(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+; \pi_\infty \otimes V) \cong H^{d-n}(\mathfrak{a}_\infty\backslash\mathfrak{g}, K_\infty^+; \pi_\infty \otimes V)^\vee.$$

See [BW00, Proposition 7.6]. So once l_0 is determined, $q_0 = (d - l_0)/2$ is automatic, since this is the unique choice for the interval $[q_0, q_0 + l_0]$ to be symmetric about $d/2$.

Example 1.23. We compute l_0 and q_0 for some groups.

(i) Let $\mathbb{G} = \mathrm{SL}_2/\mathbb{Q}$. Then $d = \dim_\mathbb{R} Y_\infty = 2$.

$$l_0 = \mathrm{rk}\,\mathrm{SL}_2(\mathbb{R}) - \mathrm{rk}\,\mathrm{SO}_2(\mathbb{R}) = 1 - 1 = 0$$
$$q_0 = (d - l_0)/2 = (2 - 0)/2 = 1$$

(ii) Let $\mathbb{G} = \mathrm{GL}_2/\mathbb{Q}$. Then $d = \dim_\mathbb{R} Y_\infty = \dim \mathcal{H}^\pm = 2$.

$$l_0 = \mathrm{rk}\,\mathrm{GL}_2(\mathbb{R}) - \mathrm{rk}\,\mathbb{R}_{>0}\,\mathrm{SO}_2(\mathbb{R}) = 2 - 2 = 0$$
$$q_0 = (d - l_0)/2 = (2 - 0)/2 = 1$$

(iii) Let $\mathbb{G}$ be the algebraic group over $\mathbb{Q}$ such that $\mathbb{G}(R) = (D \otimes R)^\times$ where $D/\mathbb{Q}$ is a definite quaternion algebra. Then $G_\infty = A_\infty K_\infty$. Thus $d = \dim_\mathbb{R} Y_\infty = 0$ and moreover:

$$l_0 = q_0 = 0.$$

(iv) Let $\mathbb{G} = U(n, F/F^+, M)$ be the algebraic group over F^+ such that

$$\mathbb{G}(R) = \{g \in \mathrm{GL}_n(F \otimes_{F^+} R) : g^\dagger M g = M\}.$$

Here F^+ is a totally real field, and F/F^+ a totally imaginary extension. The operator $(-)^\dagger$ acts on $g \in \mathrm{GL}_n(F \otimes_{F^+} R)$ by first sending $g \mapsto g^\top$, then it acts on the entries of g by applying the unique non-trivial Galois automorphism of F/F^+ on the left factor of $F \otimes_{F^+} R$. It is essentially "conjugate transpose". The matrix $M \in \mathrm{GL}_n(F)$ satisfies $M^\dagger = M$, i.e. it is Hermitian. If $F^+ = \mathbb{Q}$, $F = \mathbb{Q}(i)$, and $M = \mathrm{Id}_n$, then $\mathbb{G}(\mathbb{R})$ is the usual unitary group $U(n)(\mathbb{R})$ which is compact. Therefore, $G_\infty = K_\infty$ and this forces

$$d = l_0 = q_0 = 0.$$

An important class of (essentially) tempered representations are the (essentially) discrete series representations. By definition, all discrete series representations are fundamentally tempered, and hence discrete series representations are all cohomological.

Definition 1.24. Let $\mathbb{G}$ be a reductive group over $\mathbb{R}$. An irreducible admissible representation π of $\mathbb{G}(\mathbb{R})$ with unitary central character is called *discrete series* if all matrix coefficients lie in $L^2(Z(\mathbb{R})\backslash\mathbb{G}(\mathbb{R}))$. Otherwise, π is called *essentially discrete series* if all matrix coefficients of $\pi|_{\mathbb{G}(\mathbb{R})^1}$ lie in $L^2((Z(\mathbb{R})\cap\mathbb{G}(\mathbb{R})^1)\backslash\mathbb{G}(\mathbb{R})^1)$.

The "discrete series" representations $\mathcal{D}_{k,\mu}$ introduced previously are all essentially discrete series under this definition. The ones with unitary central character are discrete series.

It is a theorem of Harish-Chandra that a semisimple Lie group admits discrete series if and only if $l_0 = 0$ [Eme14]. A reductive Lie group admits discrete series if and only if its derived subgroup admits discrete series.

Theorem 1.25. *Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R})$ is compact modulo centre. Then the irreducible admissible representations of $\mathbb{G}(\mathbb{R})$ are all essentially discrete series.*

Proof. Every irreducible admissible representation of $\mathbb{G}(\mathbb{R})$ can be twisted to have unitary central character. After twisting, every matrix coefficient is square-integrable modulo centre, since $\mathbb{G}(\mathbb{R})$ is compact modulo centre, and hence the twisted representation is discrete series. Thus the original representation is essentially discrete series. □

Corollary 1.26. *Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R})$ is compact. Then every automorphic representation of $\mathbb{G}$ is cohomological in degree zero.*

Proof. By above theorem, if π is an automorphic representation of $\mathbb{G}$, then π_∞ is essentially discrete series, and hence π is cohomological. Since $\mathbb{G}(\mathbb{R})$ is compact, $G_\infty = K_\infty$ and so

$$d = l_0 = q_0 = 0.$$

Therefore, π appears in cohomological degree zero. □

Remark 1.27. If $\mathbb{G}$ is a reductive group such that $\mathbb{G}(\mathbb{R})$ is compact modulo centre, then its locally symmetric spaces $Y(K^\infty)$ for open compact $K^\infty \leq \mathbb{G}(\mathbb{A}^\infty)$ may have non-vanishing cohomology outside of degree zero. For example, if F is a number field, consider

$$\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m.$$

However, if $\mathbb{G}/\mathbb{Q}$ is a reductive group such that its maximal $\mathbb{Q}$-split torus is also maximal $\mathbb{R}$-split, then its only non-vanishing cohomology is indeed in degree zero. Moreover, such groups also satisfy that $\mathbb{G}(\mathbb{R})$ is compact modulo centre. Therefore, for such groups, degree zero cohomology contains all of its automorphic representations. This condition was first introduced in [Gro99, Proposition 1.4] and discussed in [Eme06b, §3.2].

2. Switching from ∞ to p

Remark 2.1. This section is based on [Eme06b, §3] and [Gro99, §4,§8].

Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$. In the classical setting over the complex numbers, we call a smooth function $\phi : \mathbb{G}(\mathbb{A}) \to \mathbb{C}$ an *automorphic form* if

(i) $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \mathbb{G}(\mathbb{Q})$ and $g \in \mathbb{G}(\mathbb{A})$.
(ii) ϕ is K_∞-finite.
(iii) ϕ is $Z(\mathfrak{g}_\mathbb{C})$-finite.

(iv) ϕ is of moderate growth, i.e. there exist $C, N > 0$ such that $|f(g)| < C\|g\|^N$ for all $g \in \mathbb{G}(\mathbb{A})$ for some adelic height function $\|-\|$ on $\mathbb{G}(\mathbb{A})$.

This is the space of automorphic forms on $\mathbb{G}$, which we denote $\mathcal{A}(\mathbb{G})$. On top of this, we may additionally fix any of the following three parameters:

(a) a *K_∞-type* (ρ, W), where $\rho : K_\infty \to \mathrm{GL}(W)$ is an irreducible, hence finite-dimensional, representation such that the vector space generated by the K_∞-translates of ϕ are isomorphic to the direct sum of a finite number of copies of W.
(b) a *level* K^∞, where K^∞ is an open compact subgroup of $\mathbb{G}(\mathbb{A}^\infty)$ such that $\phi(gk) = \phi(g)$ for all $k \in K^\infty$.
(c) a *central character* ω, where ω is a character on the centre $Z(\mathbb{A})$ of $\mathbb{G}(\mathbb{A})$ such that $\phi(zg) = \omega(z)\phi(g)$ for all $z \in Z(\mathbb{A})$ and $g \in \mathbb{G}(\mathbb{A})$.

We denote the subspace $\mathcal{A}(\mathbb{G})$ corresponding to these parameters by:

$$\mathcal{A}(\mathbb{G}, \omega, W, K^\infty) \subset \mathcal{A}(\mathbb{G}).$$

If $\mathbb{G}(\mathbb{R})$ is not compact, then it is necessary to fix a central character so that we work with the subspace $\mathcal{A}(\mathbb{G}, \omega) \subset \mathcal{A}(\mathbb{G})$ and not $\mathcal{A}(\mathbb{G})$ directly.

Proposition 2.2. *If $\mathbb{G}(\mathbb{R})$ is compact, then for functions in $\mathcal{A}(\mathbb{G})$, conditions (iii) and (iv) are implied by conditions (i) and (ii). If $\mathbb{G}(\mathbb{R})$ is compact modulo centre, then the same conclusion holds for functions in $\mathcal{A}(\mathbb{G}, \omega)$ for a fixed central character ω.*

The proposition tells us that, in particular, for groups $\mathbb{G}$ such that $\mathbb{G}(\mathbb{R})$ is compact or compact modulo centre, conditions (iii) and (iv) in the definition of an automorphic form are redundant. Hence only the algebraic conditions imposed by (i) and (ii) are relevant. This suggests that a purely algebraic definition of automorphic forms may be possible.

To simplify exposition, from now on let us assume $\mathbb{G}(\mathbb{R})$ is compact. For a fixed irreducible finite-dimensional representation (σ, V) of $\mathbb{G}(\mathbb{R})$, we can consider the space of V-valued *modular forms* $\mathcal{M}(\mathbb{G}, V)$ which consists of smooth functions $F : \mathbb{G}(\mathbb{A}) \to V$ such that

(i) $F(\gamma g) = F(g)$ for all $\gamma \in \mathbb{G}(\mathbb{Q})$ and $g \in \mathbb{G}(\mathbb{A})$.
(ii) $F(gk) = \sigma(k^{-1})F(g)$ for all $k \in \mathbb{G}(\mathbb{R})^+$ and $g \in \mathbb{G}(\mathbb{A})$.

As before, we can additionally fix a level K^∞, requiring $F(gk) = F(g)$ for all $k \in K^\infty$, and call the subspace corresponding to this parameter $\mathcal{M}(\mathbb{G}, V, K^\infty) \subset \mathcal{M}(\mathbb{G}, V)$.

Since $\mathbb{G}(\mathbb{R})$ is compact, its maximal compact subgroup K_∞ must equal itself. So there is no difference between choosing K_∞-types or $\mathbb{G}(\mathbb{R})$-representations. By choosing (σ, V) to be a K_∞-type, which is at the same time a $\mathbb{G}(\mathbb{R})$-representation, the spaces $\mathcal{M}(\mathbb{G}, V, K^\infty)$ and $\mathcal{A}(\mathbb{G}, V, K^\infty)$ are both well-defined. If $\mathbb{G}(\mathbb{R})$ is moreover connected, so that $\mathbb{G}(\mathbb{R})^+ = \mathbb{G}(\mathbb{R})$ in condition (ii) of the definition of $\mathcal{M}$, then $\mathcal{M}$ is related to $\mathcal{A}$ as follows.

Proposition 2.3. *If $\mathbb{G}(\mathbb{R})$ is compact and connected, then there is an isomorphism of complex vector spaces respecting a suitably defined level K^∞ Hecke action on both sides:*

$$\begin{aligned} \mathcal{M}(\mathbb{G}, V, K^\infty) &\to \mathrm{Hom}_{\mathbb{G}(\mathbb{R})\times K^\infty}(V^\vee, \mathcal{A}(\mathbb{G}, V^\vee, K^\infty)) \\ F &\mapsto T_F(\lambda) := \lambda \circ F. \end{aligned}$$

This induces a one-to-one correspondence between simple Hecke modules on the left and automorphic representations $\pi = \pi_\infty \otimes \pi^\infty$ of $\mathbb{G}$ such that $\pi_\infty \cong V^\vee$ and $(\pi^\infty)^{K^\infty} \neq 0$.
[Gro99, Proposition 8.5]

Why go through all the trouble to reinterpret automorphic forms? Well, the reason is because the space of functions on the left-hand side admit a cohomological description.

Let $U/\mathbb{Q}$ be an algebraic representation of $\mathbb{G}(\mathbb{Q})$. If $\mathbb{G}(\mathbb{R})$ is compact, then we have already seen that all of its automorphic representations are cohomological in degree zero, and it has no other non-vanishing cohomology. The zeroth cohomology group $H^0(Y(K^\infty), \mathcal{F}_U)$ is by definition the space of global sections of $\mathcal{F}_U$, consisting of locally constant sections:

$$\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{R}) \times \mathbb{G}(\mathbb{A}^\infty)/A_\infty^+ K_\infty^+ K^\infty \to \mathbb{G}(\mathbb{Q})\backslash U \times \mathbb{G}(\mathbb{R}) \times \mathbb{G}(\mathbb{A}^\infty)/A_\infty^+ K_\infty^+ K^\infty.$$

Indeed, these functions are constant on $\mathbb{G}(\mathbb{R})^+ \supset A_\infty^+ K_\infty^+$ and K^∞-cosets. So this space of locally constant sections consists of functions $f : \mathbb{G}(\mathbb{A}) \to U$ satisfying

(i) $f(\gamma g) = \gamma f(g)$ for all $\gamma \in \mathbb{G}(\mathbb{Q})$.
(ii) $f(gk) = f(g)$ for all $k \in \mathbb{G}(\mathbb{R})^+ K^\infty$.

Thus $H^0(Y(K^\infty), \mathcal{F}_U)$ is a $\mathbb{Q}$-vector space. If we tensor it with $\mathbb{C}$, we get the same space of functions, except with codomain $U \otimes \mathbb{C}$. The space $U \otimes \mathbb{C}$ comes with an upgraded action of $\mathbb{G}(\mathbb{C})$ which restricts to an action of $\mathbb{G}(\mathbb{R})$.

Proposition 2.4. *There is a Hecke-equivariant isomorphism:*

$$\begin{aligned} H^0(Y(K^\infty), \mathcal{F}_U) \otimes \mathbb{C} &\to \mathcal{M}(\mathbb{G}, U \otimes \mathbb{C}, K^\infty) \\ f &\mapsto F_\infty(g) := g_\infty^{-1} f(g). \end{aligned}$$

[Gro99, Proposition 8.3]

We take a step back to summarize what we have achieved so far:

(a) For groups $\mathbb{G}$ such that $\mathbb{G}(\mathbb{R})$ is compact and connected, there is a bijection between simple submodules of $\mathcal{M}(\mathbb{G}, V, K^\infty)$ and automorphic representations $\pi = \pi_\infty \otimes \pi^\infty$ such that $\pi_\infty \cong V^\vee$ and $(\pi^\infty)^{K^\infty} \neq 0$.
(b) If $\mathbb{G}(\mathbb{R})$ is compact, then $\mathcal{M}(\mathbb{G}, V, K^\infty)$ is isomorphic to $H^0(Y(K^\infty), \mathcal{F}_V)$ as modules for the level K^∞ Hecke algebra.
(c) So simple submodules of $H^0(Y(K^\infty), \mathcal{F}_V)$ correspond bijectively with the automorphic representations $\pi = \pi_\infty \otimes \pi^\infty$ such that $\pi_\infty \cong V^\vee$ and $(\pi^\infty)^{K^\infty} \neq 0$. This matches up with the statement of Matsushima's formula. Moreover, as we vary V and K^∞, we will exhaust all automorphic representations of $\mathbb{G}$. This matches up with our observation that all automorphic representations of $\mathbb{G}$ are cohomological in degree zero.

We now attempt to move from ∞ to p. Fix an irreducible finite-dimensional representation of $\mathbb{G}(\mathbb{Q}_p)$ on a $\overline{\mathbb{Q}}_p$-vector space M. Let $K^\infty = K_p K^p$ be an open compact subgroup of $\mathbb{G}(\mathbb{A}^\infty)$ that is a product of open compact subgroups $K_p \leq \mathbb{G}(\mathbb{Q}_p)$ and $K^p \leq \mathbb{G}(\mathbb{A}^{\infty,p})$. We may then define the space of M-valued *p-adic modular forms* $\mathcal{M}_p(\mathbb{G}, M, K_p K^p)$ as the space of smooth functions $F : \mathbb{G}(\mathbb{A}) \to M$ satisfying the following conditions:

(i) $F(\gamma g) = F(g)$ for all $\gamma \in \mathbb{G}(\mathbb{Q})$.
(ii) $F(gk) = F(g)$ for all $k \in \mathbb{G}(\mathbb{R})^+ K^p$.
(iii) $F(gk_p) = k_p^{-1} F(g)$ for all $k_p \in K_p$.

This space also comes from cohomology. Let $U/\mathbb{Q}$ be an algebraic representation of $\mathbb{G}(\mathbb{Q})$.

Proposition 2.5. *There is a Hecke-equivariant isomorphism:*

$$\begin{aligned} H^0(Y(K_p K^p), \mathcal{F}_U) \otimes \overline{\mathbb{Q}}_p &\to \mathcal{M}_p(\mathbb{G}, U \otimes \overline{\mathbb{Q}}_p, K_p K^p) \\ f &\mapsto F_p(g) := g_p^{-1} f(g). \end{aligned}$$

[Gro99, Proposition 8.6]

Going back to the analogy with our earlier construction, we want to find some space of p-adic automorphic forms "$\mathcal{A}_p$" so that an isomorphism of the following type holds:

$$\mathcal{M}_p(\mathbb{G}, M, K_pK^p) \xrightarrow{\sim} \mathrm{Hom}_{(\ldots)}(M^\vee, \mathcal{A}_p).$$

This hypothetical space $\mathcal{A}_p$ turns out to exist, and will be the space of locally analytic vectors in degree zero of completed cohomology $(\widetilde{H}^0)_{\mathrm{la}}$ to be constructed in the next section.

By taking the limit over open compact subgroups K_p of $\mathbb{G}(\mathbb{Q}_p)$, we can equip the space

$$\varinjlim_{K_p} \mathcal{M}_p(\mathbb{G}, M, K_pK^p)$$

with an action of $\mathbb{G}(\mathbb{Q}_p)$ via $x \cdot F(g) := xF(gx)$. The following lemma shows it is well-defined.

Lemma 2.6. *If $F \in \mathcal{M}_p(\mathbb{G}, M, K_pK^p)$ and $x \in \mathbb{G}(\mathbb{Q}_p)$, then*

$$x \cdot F \in \mathcal{M}_p(\mathbb{G}, M, xK_px^{-1}K^p).$$

Proof. Let $y \in xK_px^{-1}$ so $y = xkx^{-1}$ for some $k \in K_p$. Then $y^{-1} = xk^{-1}x^{-1}$ and

$$y^{-1}(x \cdot F)(g) = xk^{-1}x^{-1}xF(gx) = xk^{-1}F(gx) = xF(gxk) = xF(gyx) = (x \cdot F)(gy). \qquad \square$$

By taking the limit over open compact subgroups K_p of $\mathbb{G}(\mathbb{Q}_p)$, we can equip the space

$$\varinjlim_{K_p} H^0(Y(K_pK^p), \mathcal{F}_M)$$

with an action of $\mathbb{G}(\mathbb{Q}_p)$ via $x \cdot f(g) := f(gx)$. The following lemma shows it is well-defined.

Lemma 2.7. *If $f \in H^0(Y(K_pK^p), \mathcal{F}_M)$ and $x \in \mathbb{G}(\mathbb{Q}_p)$, then*

$$x \cdot f \in H^0(Y(xK_px^{-1}K^p), \mathcal{F}_M).$$

Proof. Let $y \in xK_px^{-1}$ so $y = xkx^{-1}$ for some $k \in K_p$. Then

$$(x \cdot f)(gy) = f(gyx) = f(gxk) = f(gx) = (x \cdot f)(g). \qquad \square$$

In particular, this action commutes with the isomorphisms $\mathcal{M}_p \xrightarrow{\sim} H^0$ for each fixed K_p. This action is also smooth, because the action of K_p is trivial in both cases. This will be the smooth part of certain "locally algebraic" representations introduced in the next section.

Finally, we introduce a way to view $\mathcal{M}_p$ as a cohomology group directly, and not just isomorphic to a cohomology group, as seen above.

Definition 2.8. Let $K^\infty = K_pK^p$ be a compact open subgroup of $\mathbb{G}(\mathbb{A}^\infty)$. If $M/\overline{\mathbb{Q}}_p$ is a K_p-module, consider the following cover of $Y(K_pK^p)$ with structure map:

$$(M \times (\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^+K_\infty^+))/K_pK^p \to \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^+K_\infty^+K_pK^p = Y(K_pK^p).$$

Here the action of K^∞ on M is via $m \cdot k = k_p^{-1}m$ and the action on the second factor is via right-translation. Let $\mathcal{G}_M$ denote the sheaf of locally constant sections of this cover. It is a local system on $Y(K^\infty)$.

Let $M/\overline{\mathbb{Q}}_p$ be a representation of $\mathbb{G}(\mathbb{Q}_p)$. Then it is immediate from the definitions that

$$H^0(Y(K_pK^p), \mathcal{G}_M) = \mathcal{M}_p(\mathbb{G}, M, K_pK^p).$$

This should not be confused with the following isomorphism, which is not an equality:

$$H^0(Y(K_pK^p), \mathcal{F}_M) \xrightarrow{\sim} \mathcal{M}_p(\mathbb{G}, M, K_pK^p).$$

Let $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ be any isomorphism. If $V/\mathbb{C}$ is a complex representation of $\mathbb{G}/\mathbb{C}$, then we can view V as a $\overline{\mathbb{Q}}_p$-vector space with an action of $\mathbb{G}/\overline{\mathbb{Q}}_p$ via ι. Let us call this representation M. In particular, M comes with a representation of $\mathbb{G}(\mathbb{Q}_p) \subset \mathbb{G}(\overline{\mathbb{Q}}_p)$.

Proposition 2.9. *Let $K^\infty = K_pK^p \leq \mathbb{G}(\mathbb{A}^\infty)$ be open compact. Then there is a natural isomorphism of $\pi_0 := (\mathbb{G}(\mathbb{R})/\mathbb{G}(\mathbb{R})^+)$-equivariant $\overline{\mathbb{Q}}_p$-local systems over $Y(K_pK^p)$:*

$$\mathcal{F}_M \xrightarrow{\sim} \mathcal{G}_M.$$

[Eme06b, Lemma 2.2.4]

Example 2.10. Let $\mathbb{G} = \mathrm{GL}_1/\mathbb{Q}$. This does not fit into the framework developed above because $\mathbb{G}(\mathbb{R}) = \mathbb{R}^\times$ is not compact. But I think it's a clarifying example to think about. At least, it was the first example that made sense to me. We follow [Buz04] and [Sno10].

Definition 2.11. A continuous map $\mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ is called a *Hecke character*.

Every automorphic representation of $\mathbb{G}$ is the $\mathbb{C}$-span of a Hecke character, and vice versa. In particular, automorphic representations of $\mathbb{G}$ are one-dimensional. Recall:

$$\mathbb{Q}^\times\backslash\mathbb{A}^\times = \mathbb{R}_{>0} \times \prod_p \mathbb{Z}_p^\times.$$

Let ϕ be a Hecke character. The restriction of ϕ to $\prod_p \mathbb{Z}_p^\times$ is a finite order character, as are all continuous characters from a profinite group to $\mathbb{C}^\times$. The restriction of ϕ to $\mathbb{R}_{>0}$ is a character $x \mapsto x^a$ for some real number a, as are all continuous characters from $\mathbb{R}_{>0} \to \mathbb{C}^\times$. Therefore, $\phi = \eta\kappa_\infty^a$ for some finite order character η on $\prod_p \mathbb{Z}_p^\times$ and $\kappa_\infty : \mathbb{R}_{>0} \to \mathbb{C}^\times$ is the inclusion map. Every Hecke character has this form.

Definition 2.12. A Hecke character $\phi = \eta\kappa_\infty^a$ is called *algebraic* if $a \in \mathbb{Z}$.

Definition 2.13. A continuous map $\mathbb{Q}^\times\backslash\mathbb{A}^\times \to \overline{\mathbb{Q}}_p^\times$ is called a *p-adic Hecke character*.

Let ϕ be a p-adic Hecke character. The restriction of ϕ to $\mathbb{R}_{>0}$ is trivial because the topologies on $\mathbb{R}_{>0}$ and $\overline{\mathbb{Q}}_p$ are terribly mismatched. The restriction of ϕ to $\prod_{l\neq p} \mathbb{Z}_l^\times$ is a finite order character. The restriction of ϕ to $\mathbb{Z}_p^\times$ is some continuous homomorphism.

Definition 2.14. A p-adic Hecke character is called *algebraic* if there is a compact open subgroup U of $\mathbb{Z}_p^\times$ such that $\phi|_U(x) = x^n$ for some $n \in \mathbb{Z}$.

Therefore, every algebraic p-adic Hecke character ϕ has the form $\phi = \eta\kappa_p^n$ for some integer n where η is a finite order character on $\prod_p \mathbb{Z}_p^\times$ and $\kappa_p : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times$ is the inclusion map.

Given an algebraic Hecke character $\eta\kappa_\infty^n$, we can kill the action at ∞ to get a locally constant character η, and then add back the same action at p to get $\eta\kappa_p^n$. This process is the "∞-to-p switch". It is made possible by the fact that $x \mapsto x^n$ is a character of the algebraic group $\mathbb{G}/\mathbb{Q}$, which we can then tensor over either $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$ to make sense in either setting.

In the setting of this chapter, with $\mathbb{G}$ a reductive group over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R})$ is compact and connected, the analogous algebraicity condition is the choice of a representation $U/\mathbb{Q}$ of the algebraic group $\mathbb{G}/\mathbb{Q}$. Then passing between complex automorphic forms $\mathcal{A}$ and the soon to be introduced p-adic automorphic forms $\mathcal{A}_p$ is just a matter of tracing your objects along the following identifications which have already been introduced. Let $U_\infty := U \otimes \mathbb{C}$ denote the representation $\mathbb{G}(\mathbb{R})$ and $U_p := U \otimes \overline{\mathbb{Q}}_p$ denote the representation of $\mathbb{G}(\mathbb{Q}_p)$.

$$\begin{tikzcd}
H^0(Y(K^\infty), \mathcal{F}_U) \otimes \mathbb{C} \arrow[r, "\iota:\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p"] & H^0(Y(K_pK^p), \mathcal{F}_U) \otimes \overline{\mathbb{Q}}_p \arrow[d] \\
\mathcal{M}(\mathbb{G}, U_\infty, K^\infty) \arrow[u] & \mathcal{M}_p(\mathbb{G}, U_p, K_pK^p) \arrow[d] \\
\mathrm{Hom}_{\mathbb{G}(\mathbb{R})\times K^\infty}(U_\infty^\vee, \mathcal{A}(\mathbb{G}, U_\infty^\vee, K^\infty)) \arrow[u] & \mathrm{Hom}_{(\dots)}(U_p^\vee, \mathcal{A}_p)
\end{tikzcd}$$

The top horizontal arrow denotes passage between the two spaces H^0 of locally constant functions. This corresponds to the step in the "∞ to p switch" when all the action at ∞ has been killed, so you are just left with a locally constant function, so you change your base field from $\mathbb{C}$ to $\overline{\mathbb{Q}}_p$ in anticipation of adding back the action at p.

3. Completed cohomology

Remark 3.1. This section is based on [Eme06b, §2], [Gro99, §8,§9], [ST04, §2,§3,§4], [Eme17].

Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R})$ is compact and connected. Then its automorphic representations are all essentially discrete series, hence cohomological, and they can all be found in degree zero of cohomology of some local system.

If M is a $\mathbb{G}(\mathbb{Q}_p)$-representation on a $\overline{\mathbb{Q}}_p$-vector space, then recall from before that we can form the $\overline{\mathbb{Q}}_p$-local system $\mathcal{G}_M$, and the following $\overline{\mathbb{Q}}_p$-vector space:

$$H^0(Y(K_pK^p), \mathcal{G}_M) = \mathcal{M}_p(\mathbb{G}, M, K_pK^p).$$

We want to "complete" this cohomology group. So our next immediate step should be to look for a suitable integral lattice inside it, in order to take reductions modulo a prime.

For technical reasons, it will be easier to work with vector spaces over a finite extension of $\mathbb{Q}_p$ rather than $\overline{\mathbb{Q}}_p$. So let us fix a finite extension $E/\mathbb{Q}_p$ contained inside $\overline{\mathbb{Q}}_p$, and let M be a $\mathbb{G}(\mathbb{Q}_p)$-representation on a E-vector space. If E is chosen so that $\mathbb{G}$ splits over E, then any representation of $\mathbb{G}$ on a $\overline{\mathbb{Q}}_p$-vector space descends uniquely, up to an isomorphism, to a representation of $\mathbb{G}$ on a E-vector space. Therefore, there is no loss of generality by restricting ourselves to considering only representations on E-vector spaces.

Let $K^\infty = K_pK^p$ be an open compact subgroup of $\mathbb{G}(\mathbb{A}^\infty)$. Let M_0 denote a $\mathcal{O}_E$-lattice inside M such that $K_pM_0 \subset M_0$. Then it makes sense to define the $\mathcal{O}_E$-local system $\mathcal{G}_{M_0}$ on $Y(K_pK^p)$ which is a subsheaf of $\mathcal{G}_M$. Taking global sections gives an $\mathcal{O}_E$-submodule

$$H^0(Y(K_pK^p), \mathcal{G}_{M_0}) \subset H^0(Y(K_pK^p), \mathcal{G}_M).$$

By its translation properties, a function $F : \mathbb{G}(\mathbb{A}) \to M$ in $H^0(Y(K_pK^p), \mathcal{G}_M)$ is determined by its values on any set of coset representatives for the finite quotient:

$$Y(K_pK^p) = \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbb{R})^+K_pK^p.$$

Therefore, tensoring up to E yields an isomorphism of E-vector spaces:

$$H^0(Y(K_pK^p), \mathcal{G}_{M_0}) \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} H^0(Y(K_pK^p), \mathcal{G}_M).$$

We now want to vary the level K_p. Let $\mathcal{S}$ denote the directed set of all open compact subgroups of $\mathbb{G}(\mathbb{Q}_p)$, directed downward by inclusion. Let M_0 be a separated lattice in M.

Definition 3.2. Let ϖ be a uniformizer of $\mathcal{O}_E$. Then an $\mathcal{O}_E$-module M_0 is *separated* if

$$\bigcap_{n\geq 0} \varpi^n M_0 = 0.$$

Let $\mathcal{S}_{M_0}$ denote the directed subset of $\mathcal{S}$ consisting of those open compact subgroups which preserve the lattice M_0. If G_0 is the maximal subgroup of $\mathbb{G}(\mathbb{Q}_p)$ preserving M_0, then G_0 is open compact in $\mathbb{G}(\mathbb{Q}_p)$, and $\mathcal{S}_{M_0}$ is precisely the open subgroups of G_0. It then follows immediately that $\mathcal{S}_{M_0}$ is cofinal in $\mathcal{S}$. This is the key property that we were going after. Indeed, this is what tells us that varying over open compact subgroups in $\mathcal{S}_{M_0}$ is an accurate reflection of varying over all open compact subgroups. If M_0' is another separated lattice in M, then the intersection $\mathcal{S}_{M_0} \cap \mathcal{S}_{M_0'}$ is cofinal in each of $\mathcal{S}_{M_0}$, $\mathcal{S}_{M_0'}$ and $\mathcal{S}$.

We now fix a *tame level* K^p, that is, a compact open subgroup K^p of $\mathbb{G}(\mathbb{A}^{\infty,p})$. We finally define the objects involved in our discussion of completed cohomology.

$$\widetilde{H}^0(K^p, \mathcal{G}_{M_0}) := \varprojlim_s \varinjlim_{K_p \in \mathcal{S}_{M_0}} H^0(Y(K_pK^p), \mathcal{G}_{M_0}/p^s)$$

$$\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E := \widetilde{H}^0(K^p, \mathcal{G}_{M_0}) \otimes_{\mathcal{O}_E} E$$

The inductive and projective limits are taken with respect to the obvious transition maps. This is an $\mathcal{O}_E$-module, and an E-vector space, respectively. The image of $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})$ in $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E$ is an $\mathcal{O}_E$-lattice. These objects actually have a much richer structure, but to describe them we will first need to take a detour to define many of the fundamental objects involved in locally analytic representation theory. This will be the context in which we work from now on, so it is a worthwhile investment. Henceforth let E denote any non-archimedean field with absolute value $|-|$, which includes any finite extension of $\mathbb{Q}_p$.

Definition 3.3. Let V be an E-vector space. A (non-archimedean) *semi-norm* q on V is a function $q : V \to \mathbb{R}$ such that:

(i) $q(av) = |a|q(v)$ for all $a \in E$ and $v \in V$.
(ii) $q(v + w) \leq \max(q(v), q(w))$ for all $v, w \in V$.

The E-vector space V is in particular an $\mathcal{O}_E$-module, so we may consider its $\mathcal{O}_E$-submodules.

Definition 3.4. A *lattice* in an E-vector space V is an $\mathcal{O}_E$-submodule L such that for every $v \in V$ there exists $a \in E^\times$ such that $av \in L$.
[ST04, §2]

Remark 3.5. The source [ST04, §2] imposes no other conditions in the definition of a lattice, but I think it is not enough. For instance, the current definition does not imply that the following natural map is an isomorphism of E-vector spaces: $L \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} V$.

Definition 3.6. For a lattice L in V, define its *gauge* p_L to be:

$$\begin{aligned} p_L : V &\to \mathbb{R} \\ v &\mapsto \inf_{v \in aL} |a|. \end{aligned}$$

If q is a semi-norm on V, define $\mathcal{O}_E$-submodules:

$$L(q) := \{v \in V : q(v) \leq 1\} \quad \text{and} \quad L^-(q) := \{v \in V : q(v) < 1\}.$$

Lemma 3.7. *(a) For a lattice L in V, its gauge p_L is a semi-norm.*

(b) For a semi-norm q on V, the $\mathcal{O}_E$-submodules $L(q)$ and $L^-(q)$ are lattices.
(c) For a lattice L in V: $L^-(p_L) \subset L \subset L(p_L)$.
(d) For a semi-norm q on V: $cp_{L(q)} \leq q \leq p_{L(q)}$ where $c := \sup_{|b|<1}|b|$.
[ST04, Lemma 2.1]

Let $\{L_j\}_{j\in J}$ be a non-empty family of lattices in an E-vector space V such that:
(LC1) For any $j \in J$ and $a \in E^\times$ there exists $k \in J$ such that $L_k \subset aL_j$.
(LC2) For any two $i, j \in J$ there exists $k \in J$ such that $L_k \subset L_i \cap L_j$.

Definition 3.8. The subsets $\{v + L_j\}_{v\in V, j\in J}$ form the basis of a topology on V. This is called the *locally convex topology* on V defined by the family $\{L_j\}_{j\in J}$.

Definition 3.9. A *locally convex E-vector space* is an E-vector space V equipped with a locally convex topology.

There is an alternative description of locally convex vector spaces via semi-norms. Let $\{q_i\}_{i\in I}$ be a family of semi-norms on an E-vector space V. By definition, the topology on V defined by the family $\{q_i\}_{i\in I}$ is the coarsest topology on V such that:
(i) For all $i \in I$, $q_i : V \to \mathbb{R}$ is continuous.
(ii) For all $v \in V$, the translation-by-v map $V \to V$ is continuous.

For any finitely many semi-norms $q_{i_1}, \dots, q_{i_r}$ in $\{q_i\}_{i\in I}$ and real number $\varepsilon > 0$, define:

$$V(q_{i_1}, \dots, q_{i_r}; \varepsilon) := \{v \in V : q_{i_1}(v), \dots, q_{i_r}(v) \leq \varepsilon\}.$$

Lemma 3.10. *$V(q_{i_1}, \dots, q_{i_r}; \varepsilon)$ is a lattice in V. The family of all such lattices satisfies* (LC1) *and* (LC2) *and hence defines a locally convex topology on V.*
[ST04, Lemma 3.1]

Proposition 3.11. *(a) The topology on V defined by the family of semi-norms $\{q_i\}_{i\in I}$ coincides with the locally convex topology on V defined by the family of lattices*

$$\{V(q_{i_1}, \dots, q_{i_r}; \varepsilon) : i_1, \dots, i_r \in I, \varepsilon > 0\}.$$

(b) The locally convex topology on V defined by the family of lattices $\{L_j\}_{j\in J}$ coincides with the topology defined by the family of gauges $\{p_{L_j}\}_{j\in J}$.

Proposition 3.12. *Let V be a locally convex E-vector space. The following are equivalent:*
(a) V is Hausdorff.
(b) For any non-zero $v \in V$, there exists $j \in J$ such that $v \notin L_j$.
(c) For any non-zero $v \in V$, there exists $i \in I$ such that $q_i(v) \neq 0$.

We describe an important class of locally convex vector spaces called Fréchet spaces.

Proposition 3.13. *Let V be a Hausdorff locally convex E-vector space. The following are equivalent:*
(a) V is metrizable.
(b) The topology on V can be defined by a countable family of lattices.
(c) The topology on V can be defined by a countable family of semi-norms.
[ST04, Proposition 5.1]

Definition 3.14. A locally convex E-vector space V is called an *E-Fréchet space* if it is metrizable and complete.

We describe an important class of Fréchet spaces called Banach spaces.

Definition 3.15. A semi-norm q on V is called a *norm* if $q(v) = 0$ implies $v = 0$. An E-vector space equipped with a norm, denoted $\|-\|$, is called a *normed E-vector space.*

Definition 3.16. A normed E-vector space is called an *E-Banach space* if the corresponding metric space is complete.

Indeed, an E-Banach space is an E-Fréchet space. More generally, any countable projective limit of E-Banach spaces is an E-Fréchet space.

We return to our discussion about completed cohomology. Recall that the image of $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})$ in $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E$ is an $\mathcal{O}_E$-lattice. We can thus regard $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E$ as a semi-normed space, where the semi-norm is given by the gauge of the lattice. The results of [Eme06b, §2.1] then show that $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E$ is actually an E-Banach space.

Let $M_0 \subset M_0'$ be an inclusion of separated lattices in M. This induces an injection of sheaves $\mathcal{G}_{M_0} \to \mathcal{G}_{M_0'}$. Taking global sections and some limits then induces an injection of $\mathcal{O}_E$-modules $\widetilde{H}^0(K^p, \mathcal{G}_{M_0}) \to \widetilde{H}^0(K^p, \mathcal{G}_{M_0'})$. Tensoring up to E induces a continuous map of E-Banach spaces $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E \to \widetilde{H}^0(K^p, \mathcal{G}_{M_0'})_E$.

Lemma 3.17. *If $M_0 \subset M_0'$ is an inclusion of separated lattices in M, then the induced map of E-Banach spaces $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E \to \widetilde{H}^0(K^p, \mathcal{G}_{M_0'})_E$ is a topological isomorphism.*
[Eme06b, Lemma 2.2.8]

Definition 3.18.

$$\widetilde{H}^0(K^p, \mathcal{G}_M) := \varinjlim_{M_0} \widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E.$$

The *locally convex inductive limit* of a diagram is the inductive limit of vector spaces equipped with the finest locally convex topology so that the natural inclusion maps are all continuous. Here the locally convex inductive limit is taken over the directed set of separated lattices M_0 in M, directed by inclusion.

Because the transition maps in the inductive limit are isomorphisms, for any choice of M_0, the natural inclusion maps $\widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E \to \widetilde{H}^0(K^p, \mathcal{G}_M)$ are also topological isomorphisms. Therefore, $\widetilde{H}^0(K^p, \mathcal{G}_M)$ has the structure of an E-Banach space.

If $g \in \mathbb{G}(\mathbb{Q}_p)$ and M_0 is a separated lattice in M, then gM_0 is a separated lattice in M. So $\mathbb{G}(\mathbb{Q}_p)$ acts on the directed set of separated lattices in M.

Lemma 3.19. *The action of $\mathbb{G}(\mathbb{Q}_p)$ on the directed set of separated lattices in M lifts to a continuous action of $\mathbb{G}(\mathbb{Q}_p)$ on the inductive system $\widetilde{H}^0(K^p, \mathcal{G}_M) = \varinjlim_{M_0} \widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E$.*
[Eme06b, Lemma 2.2.10]

Proof. The following map is an isomorphism of $\mathcal{O}_E$-modules:

$$H^0(Y(K_pK^p), \mathcal{G}_{M_0}) \xrightarrow{\sim} H^0(Y(g^{-1}K_pgK^p), \mathcal{G}_{g^{-1}M_0})$$
$$F(x) \mapsto g^{-1}F(xg^{-1}).$$

This induces the following isomorphism on the completed cohomology groups:

$$\widetilde{H}^0(K^p, \mathcal{G}_{M_0}) \xrightarrow{\sim} \widetilde{H}^0(K^p, \mathcal{G}_{g^{-1}M_0}).$$

Taking the inductive limit across all possible M_0 produces the required action:

$$\widetilde{H}^0(K^p, \mathcal{G}_M) \xrightarrow{\sim} \widetilde{H}^0(K^p, \mathcal{G}_M).$$

For any M_0, there is an open compact subgroup G_0 of $\mathbb{G}(\mathbb{Q}_p)$ that preserves it. This implies that the $\mathbb{G}(\mathbb{Q}_p)$ action on the inductive system over all M_0 is continuous. $\square$

So $\mathbb{G}(\mathbb{Q}_p)$ acts on the E-Banach space $\widetilde{H}^0(K^p, \mathcal{G}_M)$. The action is both continuous and admissible, the latter being a notion which we shall now define. Let us begin with a brief overview of locally analytic groups, which is required in the definition of admissibility.

Let F be a non-archimedean field, with absolute value $|-|$. We typically choose F to be a finite extension of $\mathbb{Q}_p$. For $\alpha \in \mathbb{Z}^n_{\geq 0}$, denote $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $t^\alpha := t_1^{\alpha_1} \dots t_n^{\alpha_n}$.

Definition 3.20. Let $U \subset F^n$ be open. We say that a function $g : U \to F^m$ is *locally analytic* if for all $x_0 \in U$, there exists $\varepsilon > 0$ so that for all $x \in U$ with $\|x - x_0\| \leq \varepsilon$ one has:

$$g(x) = (G_1(x - x_0), \dots, G_m(x - x_0))$$

where $G_j(t) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_{j,\alpha} t^\alpha \in F[\![t_1, \dots, t_n]\!]$ satisfies $\lim_{|\alpha| \to \infty} \varepsilon^{|\alpha|} c_{j,\alpha} = 0$ for all $j = 1, \dots, m$. The space of all such functions is denoted $\mathcal{C}^{\mathrm{la}}(U, F^m)$.

Remark 3.21. In [Sch11, §6], he extends the notion of locally analyticity to functions $U \to V$ for open subsets $U \subset F^n$ and F-Banach spaces V.

A *locally F-analytic manifold* is a pair $(X, \mathcal{A})$ where X is a Hausdorff topological space, and $\mathcal{A}$ is a maximal atlas whose charts induce isomorphisms between open sets in X and open sets in F^n for a fixed n. It is locally analytic in the sense that the transition maps between charts are locally analytic in the sense defined above. See [Sch11, §7] for details.

Definition 3.22. A *locally F-analytic group* is a locally F-analytic manifold G together with a group structure in which the multiplication map $G \times G \to G$ is locally analytic.

Let F be a finite extension of $\mathbb{Q}_p$, and E be an extension of F, complete with respect to a discrete valuation extending the discrete valuation on F.

If X is a locally F-analytic manifold, and V is a Hausdorff locally convex E-vector space, then it is possible to define more generally locally analytic functions $X \to V$. The idea is that we only consider functions $X \to V$ whose image land inside subspaces of V that are actually E-Banach spaces, and take their union. These subspaces are called BH-spaces. The space of all locally analytic functions $X \to V$ is denoted $\mathcal{C}^{\mathrm{la}}(X, V)$. For the precise definitions of these spaces, please refer to [Eme17, Definition 2.1.25] and [ST04, §10].

Definition 3.23. Let V be a Hausdorff locally convex E-vector space. We say that V is an *FH-space* if it admits a complete metric that induces a locally convex topology on V finer than its given topology. We refer to the topological vector space structure on V induced by such a metric as a *latent Fréchet space structure* on V. If this latent Fréchet space structure can be defined by a norm, so that it is in fact a latent Banach space structure, then we say that V is a *BH-space.*
[Eme17, Definition 1.1.1]

Definition 3.24. Let V be a Hausdorff locally convex E-vector space. We say that V is of *LF-type* (resp. *LB-type*) if we can write $V = \bigcup_{n=1}^\infty V_n$ for some increasing sequence $V_1 \subset V_2 \subset V_3 \subset \dots$ of *FH-spaces* (resp. *BH-spaces*).
[Eme17, Definition 1.1.9]

Definition 3.25. Let V be a locally convex E-vector space. We say that V is an *LF-space* (resp. *LB-space*) if it is isomorphic to the locally convex inductive limit of a sequence of

E-Fréchet spaces (resp. E-Banach spaces).
[Eme17, Definition 1.1.16(i)]

Definition 3.26. Let V be a Hausdorff locally convex E-vector space. We say that V is of *compact type* if it is isomorphic to the locally convex inductive limit of a sequence of locally convex E-vector spaces in which the transition maps are compact.
[Eme17, Definition 1.1.16(ii)]

Remark 3.27. I do not know the origin of the terms FH, BH, LF, and LB. However, you may find the mnemonics "FH = Fréchet, Has" (resp. "BH = Banach, Has") and "LF = Limit Fréchet" (resp. "LB = Limit Banach") to be helpful.

Definition 3.28. Let X be a Hausdorff topological space, and V be a Hausdorff locally convex E-vector space. Let $\mathcal{C}(X, V)$ denote the E-vector space of continuous V-valued functions on X, equipped with the (Hausdorff locally convex) topology of uniform convergence on compact sets.
[Eme17, Definition 2.1.2]

Proposition 3.29. *Let V be a Hausdorff locally convex E-vector space and X be a locally F-analytic manifold. Then evaluation at points of X induces a continuous injection*

$$\mathcal{C}^{\mathrm{la}}(X, V) \to \mathcal{C}(X, V).$$

It is natural in the sense that it is compatible with the functorial properties of its source and target. Moreover, this injection has dense image.
[Eme17, Proposition 2.1.26]

Definition 3.30. If X is a Hausdorff topological space, then we let $\mathcal{D}(X, E)$ denote the dual space to the locally convex E-vector space $\mathcal{C}(X, E)$. This is the space of E-valued measures on X.
[Eme17, Definition 2.2.1]

Definition 3.31. If X is a locally F-analytic manifold, then we let $\mathcal{D}^{\mathrm{la}}(X, E)$ denote the dual space to the locally convex E-vector space $\mathcal{C}^{\mathrm{la}}(X, E)$. This is the space of E-valued locally analytic distributions on X.
[Eme17, Definition 2.2.3]

The dual space to a locally convex space can be endowed with various non-canonical locally convex topologies. Frequently, we shall endow these spaces with their *strong topologies*, in which case we add the subscript "b" to emphasize this, i.e. $\mathcal{D}(X, E)_b$ or $\mathcal{D}^{\mathrm{la}}(X, E)_b$.

Proposition 3.32. *Let G be a locally compact topological group.*
(a) There is an associative product on $\mathcal{D}(G, E)_b$.
(b) If V is a Hausdorff locally convex E-vector space equipped with a continuous G-action, then we can view V_b' as a left $\mathcal{D}(G, E)_b$-module.

[Eme17, Corollary 5.1.7]

Proposition 3.33. *Let G be a locally F-analytic group.*
(a) There is an associative product on $\mathcal{D}^{\mathrm{la}}(G, E)_b$.
(b) If V is a Hausdorff LF-space over E equipped with a locally analytic action of G, then we can view V_b' as a left $\mathcal{D}^{\mathrm{la}}(G, E)_b$-module.

[Eme17, Corollary 5.1.9]

We have yet to define locally analytic representations, but when we do, we will also wish to consider a subset of them which are admissible, and the definition of admissibility is a finiteness condition on modules over the distribution algebra.

Definition 3.34. Let G be a locally F-analytic group. A continuous G-action on an E-Banach space V is an *admissible continuous representation* of G if V'_b is finitely generated as a left $\mathcal{D}(H, E)_b$-module for one (and hence every) compact open subgroup H of G. [Eme17, Proposition-Definition 6.2.3]

Definition 3.35. Let G be a locally F-analytic group, and Γ a locally compact group. Let V be a Hausdorff locally convex E-vector space equipped with a topological action of $G \times \Gamma$. We say that V is an *admissible continuous representation* of $G \times \Gamma$ if:

(i) For each open compact subgroup H of Γ, the closed subspace V^H in V is an E-Banach space, and the G-action on V^H is an admissible continuous G-representation.
(ii) The Γ-action on V is *strictly smooth*, that is, the natural map $\varinjlim_H V^H \to V$ is a topological isomorphism, where the locally convex inductive limit is taken over all open compact subgroups H of Γ.

[Eme17, Definition 7.2.1]

We return to our discussion about completed cohomology.

Theorem 3.36. *Let $\pi_0 := \mathbb{G}(\mathbb{R})/\mathbb{G}(\mathbb{R})^+$. The group $\pi_0 \times \mathbb{G}(\mathbb{Q}_p)$ acts on the E-Banach space $\widetilde{H}^0(K^p, \mathcal{G}_M)$ via an admissible continuous representation.*
[Eme06b, Theorem 2.2.11]

So far the tame level K^p has been kept fixed. We now describe a way to package all of the tame levels together, into a single module.

Definition 3.37.
$$\widetilde{H}^0(\mathcal{G}_M) := \varinjlim_{K^p} \widetilde{H}^0(K^p, \mathcal{G}_M).$$
Here the locally convex inductive limit is taken over the directed set of all compact open subgroups of $\mathbb{G}(\mathbb{A}^{\infty,p})$, directed downward by inclusion.

The transition maps here are just the obvious inclusions of functions. After passing to the inductive limit of all tame levels, we can upgrade our original representation of $\pi_0 \times \mathbb{G}(\mathbb{Q}_p)$ to a representation of the bigger group:
$$\pi_0 \times \mathbb{G}(\mathbb{A}^\infty) = \pi_0 \times \mathbb{G}(\mathbb{Q}_p) \times \mathbb{G}(\mathbb{A}^{\infty,p}).$$
Thus $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$ can be written as the product $G \times \Gamma$ of a locally $\mathbb{Q}_p$-analytic group $G = \mathbb{G}(\mathbb{Q}_p)$ and a locally compact group $\Gamma = \pi_0 \times \mathbb{G}(\mathbb{A}^{\infty,p})$.

Theorem 3.38.
(a) The group $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$ acts on $\widetilde{H}^0(\mathcal{G}_M)$ via an admissible continuous representation.
(b) For each compact open subgroup K^p of $\mathbb{G}(\mathbb{A}^{\infty,p})$, there are natural isomorphisms of admissible continuous $\pi_0 \times \mathbb{G}(\mathbb{Q}_p)$-representations
$$\widetilde{H}^0(K^p, \mathcal{G}_M) \xrightarrow{\sim} \widetilde{H}^0(\mathcal{G}_M)^{K^p}.$$
[Eme06b, Theorem 2.2.16]

Definition 3.39. If M is the trivial representation of $\mathbb{G}(\mathbb{Q}_p)$ on E, then define:

$$\widetilde{H}^0 := \widetilde{H}^0(\mathcal{G}_M).$$

Theorem 3.40. *If M is any finite-dimensional algebraic representation of $\mathbb{G}$ defined over E, then there is a natural isomorphism of admissible continuous $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-representations:*

$$\widetilde{H}^0(\mathcal{G}_M) \to \widetilde{H}^0 \otimes_E M.$$

The action of $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty) = \pi_0 \times \mathbb{G}(\mathbb{Q}_p) \times \mathbb{G}(\mathbb{A}^{\infty,p})$ on the target is via the diagonal action of $\mathbb{G}(\mathbb{Q}_p)$ and the action of $\pi_0 \times \mathbb{G}(\mathbb{A}^{\infty,p})$ on the first factor.
[Eme06b, Theorem 2.2.17]

We want to pass from $\widetilde{H}^0$ to the subspace "$\widetilde{H}^0_{\mathrm{la}}$" of locally analytic vectors, so that our admissible continuous representations upgrade to locally analytic representations. We define now what it means to have an (essentially) locally analytic representation.

Definition 3.41. Let G be a locally F-analytic group. Let V be a continuous representation of G on a locally convex E-vector space. A vector $v \in V$ is said to be a *locally analytic vector* if the orbit map $g \mapsto gv$ belongs to $\mathcal{C}^{\mathrm{la}}(G, V)$. Let V_{la} denote the space of locally analytic vectors in V. It is stable under the action of G.

Remark 3.42. We would like to give V_{la} the structure of a locally convex E-vector space so that the map $V \mapsto V_{\mathrm{la}}$ at least preserves the category it is defined on. This is done by defining V_{la} as the locally convex inductive limit of some spaces of analytic vectors. Then the image of the natural map $V_{\mathrm{la}} \to V$ is precisely the space of vectors having locally analytic orbit map as defined above. See [Eme17, Definition 3.5.3] for a careful treatment of this.

Definition 3.43. Let V be a locally convex E-vector space. It is called *barrelled* if every closed lattice in V is open.

Definition 3.44. Let G be a locally F-analytic group. Let V be a barrelled Hausdorff locally convex E-vector space, equipped with a continuous action of G. Then we say that V is a *locally analytic representation* of G if the natural map $V_{\mathrm{la}} \to V$ is a bijection.
[Eme17, Definition 3.6.9]

Definition 3.45. If G is a locally F-analytic group, and if V is a locally convex E-vector space of compact type equipped with a locally analytic representation of G, then V is *admissible* if V'_b is a coadmissible module with respect to the natural $\mathcal{D}^{\mathrm{la}}(H, E)_b$-module structure on V'_b for some (and hence every) compact open subgroup H of G.
[Eme17, Definition 6.1.1, Corollary 6.1.22]

If G is a locally F-analytic group, E is discretely valued, and H is a compact open subgroup of G, then $\mathcal{D}^{\mathrm{la}}(H, E)_b$ is a *Fréchet-Stein algebra* [Eme17, Corollary 5.3.19]. For these algebras, there is a notion of *coadmissible* modules over them, which involve being finitely generated in some sense together with other nice properties [Eme17, Definition 1.2.8].

Definition 3.46. Let G be a locally F-analytic group, whose centre Z is topologically finitely generated. (This is so that the character variety $\widehat{Z}$ is representable by a quasi-Stein rigid analytic space over F.) Let V be a locally convex E-vector space of compact type, equipped with a locally analytic representation of G. We say that V is an *essentially admissible* locally analytic representation of G if the following conditions are satisfied:

(i) The contragredient Z-action on V_b' extends to a topological $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$-module structure on V_b'. (Such an extension is unique, if it exists.)
(ii) The dual V_b' is a coadmissible module when endowed with its natural module structure over the nuclear Fréchet algebra $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)\widehat{\otimes}_E \mathcal{D}^{\mathrm{la}}(H, E)_b$ for some (and hence every) compact open subgroup H of G.

[Eme17, Definition 6.4.9]

Proposition 3.47. *Any admissible locally analytic representation of G is an essentially admissible locally analytic representation of G.*
[Eme17, Proposition 6.4.10]

Let G be a locally F-analytic group, and Γ a locally compact group. Then we want to define the same notions for representations of $G \times \Gamma$, as before. If V is a Hausdorff locally convex E-vector space equipped with a topological action of $G \times \Gamma$, we let V_{la} denote the space of G-locally analytic vectors attached to V. Then V_{la} comes equipped with a topological $G \times \Gamma$-action, uniquely determined by the requirement that the natural continuous injection $V_{\mathrm{la}} \to V$ should be $G \times \Gamma$-equivariant.

Definition 3.48. A topological action of $G \times \Gamma$ on a Hausdorff locally convex E-vector space V is a *locally analytic representation* of $G \times \Gamma$ if V is barrelled, the natural map $V_{\mathrm{la}} \to V$ is a bijection, and the Γ-action on V is strictly smooth.
[Eme17, Definition 7.2.3]

Definition 3.49. Let V be a Hausdorff locally convex E-vector space with a locally analytic representation of $G \times \Gamma$. We say that V is an *(essentially) admissible* locally analytic representation of $G \times \Gamma$ if for each open compact subgroup H of Γ, the closed subspace V^H in V is an (essentially) admissible locally analytic representation of G.
[Eme17, Definition 7.2.7]

The reason we are defining essentially admissible representations now is because when we define the locally analytic Jacquet functor in the next section, this will be a natural category to define it on, since the functor will send this category back into itself. For now, we return to our discussion about completed cohomology.

Theorem 3.50.
(a) The space $\widetilde{H}^0_{\mathrm{la}}$ is an admissible locally analytic representation of $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$.
(b) For any compact open subgroup K^p of $\mathbb{G}(\mathbb{A}^{\infty,p})$, the natural map

$$\widetilde{H}^0(K^p)_{\mathrm{la}} \to (\widetilde{H}^0_{\mathrm{la}})^{K^p}$$

is a $\pi_0 \times \mathbb{G}(\mathbb{Q}_p)$-equivariant isomorphism.

[Eme06b, Theorem 2.2.22]

The following example should hopefully motivate everything that we have done so far.

Example 3.51. Let us calculate $\widetilde{H}^0_{\mathrm{la}}$ for $\mathbb{G} = \mathrm{GL}_1/\mathbb{Q}$.

Fix a tame level K^p in $\prod_{q \neq p} \mathbb{Z}_q^\times$. Let E be a finite extension of $\mathbb{Q}_p$, and $M = E$ be equipped with the trivial action of the locally $\mathbb{Q}_p$-analytic group $\mathbb{G}(\mathbb{Q}_p) = \mathbb{Q}_p^\times$. Then $M_0 = \mathcal{O}_E$ is a separated lattice in M. The subgroups $K_{p,r} = 1 + p^r\mathbb{Z}_p$ are cofinal in the directed set $\mathcal{S}_{M_0}$. Therefore, one has:

$$\widetilde{H}^0(K^p, \mathcal{G}_{M_0}) := \varprojlim_s \varinjlim_r H^0(Y(K_{p,r}K^p), \mathcal{G}_{M_0}/p^s).$$

Indeed, the argument $H^0(\dots)$ is just the space of functions $F : \mathbb{A}^\times \to \mathcal{O}_E/p^s$ such that:

(i) $F(\gamma g) = F(g)$ for all $\gamma \in \mathbb{Q}^\times$.
(ii) $F(gk) = F(g)$ for all $k \in \mathbb{R}_{>0}K^p$.
(iii) $F(gk_p) = F(g)$ for all $k_p \in K_{p,r}$.

After passing through the inverse limit, we get functions with codomain $\mathcal{O}_E$. This is because $\mathcal{O}_E = \varprojlim_s \mathcal{O}_E/\mathfrak{p}^s$ where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_E$. But $p\mathcal{O}_E = \mathfrak{p}^e$ for some integer $e \geq 1$ and hence $\{p^s\mathcal{O}_E\}_{s\geq 1} \subset \{\mathfrak{p}^s\}_{s\geq 1}$ forms a cofinal system. So after passing to both the direct and inverse limits, we get the space of functions $F : \mathbb{A}^\times \to \mathcal{O}_E$ such that:

(i) $F(\gamma g) = F(g)$ for all $\gamma \in \mathbb{Q}^\times$.
(ii) $F(gk) = F(g)$ for all $k \in \mathbb{R}_{>0}K^p$.
(iii) For all $s \geq 1$, there exists $r \geq 1$, such that for all $k_p \in K_{p,r}$:

$$F(gk_p) \equiv F(g) \pmod{p^s\mathcal{O}_E}.$$

Condition (iii) is just saying that $F|_{\mathbb{Q}_p^\times}$ is continuous. Let $(\widehat{\mathbb{Z}}^{(p)})^\times := \Pi_{q\neq p}\mathbb{Z}_q^\times$ then

$$\begin{aligned}\widetilde{H}^0(K^p, \mathcal{G}_{M_0}) &= \{F : \mathbb{A}^\times \to \mathcal{O}_E : F \text{ satisfies (i), (ii), and (iii)}\}\\ &= \{F : \mathbb{R}_{>0} \times \Pi_q\mathbb{Z}_q^\times \to \mathcal{O}_E : F|_{\mathbb{Z}_p^\times} \text{ is continuous, } F|_{\mathbb{R}_{>0}(\widehat{\mathbb{Z}}^{(p)})^\times} \text{ is } \mathbb{R}_{>0}K^p\text{-fixed}\}.\end{aligned}$$

Since the action of $\mathbb{G}(\mathbb{Q}_p)$ on separated lattices in M is trivial, that is, $gM_0 = M_0$ for every separated lattice M_0 in M, we obtain a $\mathbb{G}(\mathbb{Q}_p)$-equivariant isomorphism:

$$\widetilde{H}^0(K^p, \mathcal{G}_M) := \varinjlim_{M_0} \widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E = \widetilde{H}^0(K^p, \mathcal{G}_{M_0})_E$$

where $M_0 = \mathcal{O}_E$ is the lattice that we previously selected. Therefore, taking the locally convex inductive limit over all tame levels K^p, one obtains:

$$\begin{aligned}\widetilde{H}^0(\mathcal{G}_M) &= \{F : \mathbb{R}_{>0} \times \Pi_q\mathbb{Z}_q^\times \to \mathcal{O}_E : F|_{\mathbb{Z}_p^\times} \text{ is continuous, } F|_{\mathbb{R}_{>0}(\widehat{\mathbb{Z}}^{(p)})^\times} \text{ is locally constant}\}\\ &= \mathcal{C}(\mathbb{Z}_p^\times, E) \otimes_E \mathcal{C}^{\mathrm{sm}}((\widehat{\mathbb{Z}}^{(p)})^\times, E).\end{aligned}$$

The action of $g \in \mathbb{G}(\mathbb{Q}_p)$ on a function $F : \mathbb{A}^\times \to E$ in $\widetilde{H}^0 := \widetilde{H}^0(\mathcal{G}_M)$ is via the formula:

$$g \cdot F(x) := gF(xg) = F(xg).$$

The locally analytic vectors in $\widetilde{H}^0$ are the ones for which the orbit maps $g \mapsto F(xg)$ are locally analytic, that is, belong to $\mathcal{C}^{\mathrm{la}}(\mathbb{Q}_p^\times, E)$. However, choosing $x = 1$, this implies that the function $F|_{\mathbb{Q}_p^\times}$ itself must be locally analytic, and hence so must $F|_{\mathbb{Z}_p^\times}$. We conclude:

$$\widetilde{H}^0_{\mathrm{la}} := \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^\times, E) \otimes_E \mathcal{C}^{\mathrm{sm}}((\widehat{\mathbb{Z}}^{(p)})^\times, E).$$

Remark 3.52. The above example is true more generally. For a general $\mathbb{G}$, one can show that $\widetilde{H}^0$ is the space of continuous E-valued functions on $\mathbb{G}(\mathbb{A})$ that are locally constant on $\mathbb{G}(\mathbb{A}^{\infty,p})$-cosets and which are invariant under the left-translation action of $\mathbb{G}(\mathbb{Q})$. The functions in the subspace $\widetilde{H}^0_{\mathrm{la}}$ are then just the functions in $\widetilde{H}^0$ which additionally satisfy that they are locally analytic on $\mathbb{G}(\mathbb{Q}_p)$-cosets. See [Eme06b, p. 55] for this discussion.

We wish to also consider the subspace of "locally algebraic vectors" in $\widetilde{H}^0$. This will be contained in the subspace of locally analytic vectors $\widetilde{H}^0_{\mathrm{la}}$. Let us take a brief moment to introduce the notions of locally algebraic vectors and representations.

Let $\mathbb{G}$ be a connected reductive group over F. Let G be an open subgroup of $\mathbb{G}(F)$. Let $\mathcal{R}$ be the category of finite dimensional algebraic representations of $\mathbb{G}$ on E-vector spaces. Since $\mathbb{G}$ is reductive, the category $\mathcal{R}$ is semi-simple and abelian.

Definition 3.53. Let V be a representation of G on an E-vector space, and let M denote an object of $\mathcal{R}$. We say that a vector $v \in V$ is *locally M-algebraic* if there exists an open subgroup H of G, a natural number n, and an H-equivariant homomorphism $M^n \to V$ whose image contains the vector v. We say that V is a *locally M-algebraic representation* if every vector of V is locally M-algebraic.
[Eme17, Definition 4.2.1]

Definition 3.54. Let V be a representation of G on an E-vector space, and let M denote an object of $\mathcal{R}$. Let $V_{M\text{-lalg}}$ be the G-invariant subspace of locally M-algebraic vectors of V.
[Eme17, Proposition-Definition 4.2.2]

Definition 3.55. Let V be a representation of G on an E-vector space.

(a) A vector $v \in V$ is *locally algebraic* if it is locally M-algebraic for some object M of $\mathcal{R}$.
(b) The set V_{lalg} of all locally algebraic vectors of V forms a G-invariant subspace.
(c) We say that V is a *locally algebraic representation* of G if $V_{\text{lalg}} = V$.

[Eme17, Proposition-Definition 4.2.6]

Let $\widehat{\mathbb{G}}$ denote a set of isomorphism class representatives for the irreducible objects of $\mathcal{R}$. We remark crucially that the objects of $\widehat{\mathbb{G}}$ need not be absolutely irreducible.

Proposition 3.56. *Let V be a representation of G on an E-vector space. Then the following natural map is an isomorphism of G-representations:*

$$\bigoplus_{M \in \widehat{\mathbb{G}}} V_{M\text{-lalg}} \to V_{\text{lalg}}.$$

[Eme17, Corollary 4.2.7]

Proposition 3.57. *Let V be an irreducible locally algebraic representation of G. Then there exists*

(i) an element M of $\widehat{\mathbb{G}}$, for which we set $B := \operatorname{End}_{\mathbb{G}}(M)$, and
(ii) an irreducible smooth representation of G on a right B-module U, where irreducibility is defined with respect to representations over B,

such that V is isomorphic to $U \otimes_B M$. Conversely, given such a M and U, the tensor product $U \otimes_B M$ is an irreducible locally algebraic representation of G over E.
[Eme17, Proposition 4.2.8]

Definition 3.58. If V is a locally algebraic representation of G that becomes admissible as a locally analytic representation when equipped with its finest convex topology, then we say that V (equipped with its finest convex topology) is an *admissible locally algebraic representation* of G. If V is furthermore locally M-algebraic for some $M \in \mathcal{R}$, then we say that V is an admissible locally M-algebraic representation.
[Eme17, Definition 6.3.9]

Definition 3.59. If V is a representation of $G \times \Gamma$ that is locally algebraic as a representation of G, and that becomes admissible as a locally analytic representation of when equipped with its finest convex topology, then we say that V (equipped with its finest convex topology) is an *admissible locally algebraic representation* of $G \times \Gamma$. If V is furthermore locally M-algebraic for some $M \in \mathcal{R}$, then we say that V is an admissible locally M-algebraic representation. [Eme17, Definition 7.2.15]

Proposition 3.60. *Let V be an admissible locally M-algebraic representation of $G \times \Gamma$ for some object $M \in \mathcal{R}$. Let $B := \mathrm{End}_{\mathbb{G}}(M)$. Then there exists an admissible smooth representation of $G \times \Gamma$ on a right B-module U (equipped with its finest convex topology) such that V is isomorphic to $U \otimes_B M$. The $G \times \Gamma$-action on the tensor product is induced by the diagonal action of G, and the action of Γ on the first factor. Conversely, any such tensor product is an admissible locally M-algebraic representation of $G \times \Gamma$.*
[Eme17, Proposition 7.2.16]

We return to the discussion of completed cohomology. Let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R})$ is compact and connected. Let E be a finite extension of $\mathbb{Q}_p$ over which $\mathbb{G}$ splits; this guarantees that any irreducible representation of $\mathbb{G}$ on an E-vector space is *absolutely irreducible*. Let M be a finite-dimensional algebraic representation of $\mathbb{G}(\mathbb{Q}_p)$ on an E-vector space. Let $M^\vee$ denote the contragredient representation to M. As we pass to the inductive limit of open compact subgroups K^∞ of $\mathbb{G}(\mathbb{A}^\infty)$ shrinking to the identity, we get the following admissible smooth representation of $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$:

$$H^0(\mathcal{G}_M) := \varinjlim_{K^\infty} H^0(Y(K^\infty), \mathcal{G}_M).$$

If M is irreducible, then it is absolutely irreducible, so $\mathrm{End}_{\mathbb{G}}(M) = E$. Let

$$H^0(\mathcal{G}_M) \otimes_E M^\vee$$

be equipped with an action of $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$ induced by the diagonal action of $\mathbb{G}(\mathbb{Q}_p)$ and the action of $\pi_0 \times \mathbb{G}(\mathbb{A}^{\infty,p})$ on the left factor. We have already seen that this is a locally $M^\vee$-algebraic representation. The following result tells us that this is exactly the subspace of locally $M^\vee$-algebraic vectors in $\widetilde{H}^0$.

Proposition 3.61. *The following natural map is an isomorphism:*

$$H^0(\mathcal{G}_M) \otimes_E M^\vee \to \widetilde{H}^0_{M^\vee\text{-lalg}}.$$

[Eme06b, Corollary 2.2.25]

Factoring locally algebraic representations into a tensor product of a smooth representation with an algebraic representation is an useful description to have in hand. Later when we define the locally analytic Jacquet functor, we will see that it acts on locally algebraic representations via the smooth Jacquet functor on the smooth factor, and picking out the highest weight vector on the algebraic representation.

In the above setup, there is a natural $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-equivariant map:

$$H^0(\mathcal{G}_M) \otimes_E M^\vee \to \widetilde{H}^0_{M^\vee\text{-lalg}} \subset \widetilde{H}^0.$$

induced by the inclusion of $\widetilde{H}^0_{M^\vee\text{-lalg}}$ into $\widetilde{H}^0$.

Lemma 3.62. *This is equivalent to giving a $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-equivariant map:*

$$H^0(\mathcal{G}_M) \to (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}}.$$

Proof. Let $G \times \Gamma := \pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$. By tensor-hom adjunction,

$$\begin{aligned}
&\mathrm{Hom}_{G\times\Gamma}(H^0(\mathcal{G}_M) \otimes_E M^\vee, (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}} \otimes_E M^\vee) \\
&= \mathrm{Hom}_{G\times\Gamma}(H^0(\mathcal{G}_M), \mathrm{Hom}(M^\vee, (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}} \otimes_E M^\vee)) \\
&= \mathrm{Hom}_{G\times\Gamma}(H^0(\mathcal{G}_M), \mathrm{Hom}(M^\vee, (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}} \otimes_E M^\vee)_{\mathrm{sm}}) \\
&= \mathrm{Hom}_{G\times\Gamma}(H^0(\mathcal{G}_M), (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}}).
\end{aligned}$$

The second equality is because $H^0(\mathcal{G}_M)$ is smooth, and hence has image in the smooth vectors of the target. Since $(M \otimes_E \widetilde{H}^0)_{\mathrm{sm}}$ is smooth by definition, the third equality is a consequence of the following calculation done just before [Eme17, Proposition 4.2.4]:

$$\mathrm{Hom}(M^\vee, (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}} \otimes_E M^\vee)_{\mathrm{sm}} = (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}}.$$

Therefore, giving the latter map is equivalent to giving a map:

$$H^0(\mathcal{G}_M) \otimes_E M^\vee \to (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}} \otimes_E M^\vee.$$

However, by [Eme17, Proposition 4.2.4], there is an isomorphism:

$$(M \otimes_E \widetilde{H}^0)_{\mathrm{sm}} \otimes_E M^\vee \xrightarrow{\sim} \widetilde{H}^0_{M^\vee\text{-lalg}}.$$

This completes the proof. □

Let $\mathfrak{g}$ denote the Lie algebra of $G := \mathbb{G}(\mathbb{Q}_p)$. The functors $(-)_{\mathrm{sm}}$ and $((-)_{\mathrm{la}})^{\mathfrak{g}}$ induce the same subspaces on G-representations [Eme17, Corollary 4.1.6]. More generally, one can show that their derived functors are naturally isomorphic [Eme06b, Theorem 1.1.13]. Thus

$$\begin{aligned}
H^0(\mathcal{G}_M) \to (M \otimes_E \widetilde{H}^0)_{\mathrm{sm}} &= ((M \otimes_E \widetilde{H}^0)_{\mathrm{la}})^{\mathfrak{g}} \\
&\xrightarrow{\sim} (M \otimes_E \widetilde{H}^0_{\mathrm{la}})^{\mathfrak{g}} \qquad \text{([Eme17, Proposition 3.6.15])} \\
&= \mathrm{Hom}_{\mathfrak{g}}(M^\vee, \widetilde{H}^0_{\mathrm{la}}).
\end{aligned}$$

Therefore, it is equivalent to give either one of the following maps:

(i) $H^0(\mathcal{G}_M) \otimes_E M^\vee \to \widetilde{H}^0$.
(ii) $H^0(\mathcal{G}_M) \to \mathrm{Hom}_{\mathfrak{g}}(M^\vee, \widetilde{H}^0_{\mathrm{la}})$.

In fact, map (ii) is an isomorphism, and this follows from its origins as the edge map of a spectral sequence, which we shall see below. Note that map (ii) identifies cohomology of a local system with some maps into $\widetilde{H}^0_{\mathrm{la}}$. This is analogous to the same identification for classical automorphic forms, with $\widetilde{H}^0_{\mathrm{la}}$ replaced by $\mathcal{A}(\dots)$. In light of this analogy, it is sensible to call the $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-representation $\widetilde{H}^0_{\mathrm{la}}$ the space of *p-adic automorphic forms*.

Proposition 3.63. *The locally algebraic irreducible closed $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-subrepresentations of $\widetilde{H}^0$, equivalently just the irreducible closed $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-subrepresentations of $\widetilde{H}^0_{\mathrm{lalg}}$, are precisely the "weight-transferred-to-p" classical automorphic representations.*
[Eme06b, Proposition 3.2.4]

The precise meaning of "weight-transferred-to-p" is given in [Eme06b, Definition 3.1.5]. He calls them *classical p-adic automorphic representations.*

Example 3.64. Let us calculate $\widetilde{H}^0_{\mathrm{lalg}}$ for $\mathbb{G} = \mathrm{GL}_1/\mathbb{Q}$. Let $\Gamma := \{\pm 1\} \times (\mathbb{A}^{\infty,p})^\times$.

Let E be any finite extension of $\mathbb{Q}_p$. Clearly, $\mathbb{G}$ splits over E. Recall that the irreducible finite-dimensional algebraic representations of $G := \mathbb{G}(\mathbb{Q}_p)$ on E-vector spaces are precisely the one-dimensional representations given by characters $\chi_k : \mathbb{Q}_p^\times \to E^\times$ for some integer k such that $\chi_k(x) = x^k$. Fix some integer k, and consider $\widetilde{H}^0_{\chi_k\text{-lalg}}$. If we understand these spaces, then recall we understand $\widetilde{H}^0_{\mathrm{lalg}}$, since:

$$\widetilde{H}^0_{\mathrm{lalg}} = \bigoplus_{k \in \mathbb{Z}} \widetilde{H}^0_{\chi_k\text{-lalg}}.$$

One of our calculations earlier showed that there is an isomorphism of $G \times \Gamma$-representations:

$$H^0(\mathcal{G}_{\chi_k}) \otimes_E \chi_k \xrightarrow{\sim} \widetilde{H}^0_{\chi_k\text{-lalg}}.$$

The space $H^0(\mathcal{G}_{\chi_k})$ is a direct sum of one-dimensional $G \times \Gamma$-subrepresentations, since it is the inductive limit of finite-dimensional representations on which the action of $G \times \Gamma$ factors through a finite abelian quotient. In particular, one has:

$$H^0(\mathcal{G}_{\chi_k}) = \bigoplus_{\eta} E\eta$$

where η varies over characters $\eta : \mathbb{A}^\times \to E^\times$ such that η is trivial on $\mathbb{Q}^\times \mathbb{R}_{>0} K_p K^p$ for some open compact subgroup $K_p K^p$ of $(\mathbb{A}^\infty)^\times$. Therefore, putting everything together:

$$\widetilde{H}^0_{\mathrm{lalg}} = \bigoplus_k \widetilde{H}^0_{\chi_k\text{-lalg}} = \bigoplus_k \bigoplus_\eta E\eta\chi_k.$$

This is precisely the span of the algebraic p-adic Hecke characters previously defined.

Finally, we return to the task of showing that the following map is an isomorphism:

$$H^0(\mathcal{G}_M) \to \mathrm{Hom}_{\mathfrak{g}}(M^\vee, \widetilde{H}^0_{\mathrm{la}}).$$

This is a consequence of the following theorem, from which the map comes from.

Theorem 3.65. *For all $n \geq 0$, there is a $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-equivariant map*

$$H^n(\mathcal{G}_M) \to \mathrm{Hom}_{\mathfrak{g}}(M^\vee, \widetilde{H}^n_{\mathrm{la}})$$

which is the edge map of a $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$-equivariant spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathfrak{g}}(M^\vee, \widetilde{H}^j_{\mathrm{la}}) \Rightarrow H^{i+j}(\mathcal{G}_M).$$

[Eme06b, Corollary 2.2.18]

Since $\mathbb{G}(\mathbb{R})$ is compact, $\widetilde{H}^j = 0$ for $j \neq 0$, so $E_2^{i,j} = 0$ unless $j = 0$. So only the middle row of the following illustration of the second page of the spectral sequence is possibly non-zero.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & E_2^{-2,1} \arrow[drr] & E_2^{-1,1} & E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & \cdots \\
\cdots & E_2^{-2,0} & E_2^{-1,0} & E_2^{0,0} \arrow[drr] & E_2^{1,0} & E_2^{2,0} & \cdots \\
\cdots & E_2^{-2,-1} & E_2^{-1,-1} & E_2^{0,-1} & E_2^{1,-1} & E_2^{2,-1} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By the direction of the differentials, the spectral sequence has already converged everywhere on the second page, so choosing $i = j = 0$, we see that $\mathrm{Hom}_{\mathfrak{g}}(M^\vee, \widetilde{H}^0_{\mathrm{la}})$ is one of the graded pieces of $H^0(\mathcal{G}_M)$. For $n \geq 0$, the graded pieces of $H^n(\mathcal{G}_M)$ are $E_2^{i,j}$ such that $i + j = n$. Since we have a first quadrant spectral sequence, $i + j = 0$ implies $i = j = 0$. Therefore, we have found that $H^0(\mathcal{G}_M) = \mathrm{Hom}_{\mathfrak{g}}(M^\vee, \widetilde{H}^0_{\mathrm{la}})$. This is it.

4. Emerton's Jacquet functor

Let F be a non-archimedean locally compact field. Let $\mathbb{G}$ be a reductive algebraic group defined over F. Let S be a maximal F-split torus in $\mathbb{G}$, and let $\Phi_F := \Phi(\mathbb{G}, S)$ be the set of roots of $\mathbb{G}$ relative to F. This is a subset of $X(S) \otimes \mathbb{R}$ where $X(S) := \mathrm{Hom}(S, \mathbb{G}_m)$ is the group of algebraic characters of S. Together, $(\Phi_F, X(S) \otimes \mathbb{R})$ defines a root system of $\mathbb{G}$ relative to F defined by the maximal F-split torus S.

Choose a set of positive roots $\Phi_F^+ \subset \Phi_F$, which determines a unique minimal parabolic subgroup P containing S. We have that $P = MN$ is a Levi decomposition for P. Let $\Delta_F \subset \Phi_F^+$ be a set of simple roots for this ordering. If $\Theta \subset \Delta_F$, let $S_\Theta = \bigcap_{\alpha \in \Theta} \ker(\alpha)$. The standard parabolic F-subgroup defined by Θ is then the subgroup P_Θ generated by the centralizer $C(S_\Theta)$ of S_Θ and N. This can alternatively be written as the semidirect product $P_\Theta = C(S_\Theta) N_\Theta$ where $N_\Theta = R_u(P_\Theta)$ has Lie algebra $\sum \mathfrak{g}_\alpha$ where α runs over roots in Φ_F^+ that are not linear combinations of roots in Θ. In this notation, $P = P_\varnothing$. See discussion in [Cas95, §1] and [Bor66, §6.5] for more details.

Theorem 4.1. *Every parabolic F-subgroup of $\mathbb{G}$ is conjugate over F to one and only one standard parabolic F-subgroup.*
[Bor66, §6]

Remark 4.2. It should not be surprising that we are only considering the root system and ignoring the coroot system, because parabolic subgroups of $\mathbb{G}$ are determined by the parabolic subgroups of $\mathbb{G}^{\mathrm{der}}$. For example, consider the maps

$$\mathbb{G}^{\mathrm{der}} \to \mathbb{G} \to \mathbb{G}/Z(\mathbb{G}).$$

Since parabolic subgroups contain the center, the second map induces a bijection on parabolic subgroups of $\mathbb{G}$ and $\mathbb{G}/Z(\mathbb{G})$. The composition of the two maps is a central isogeny, which identifies the root systems in $\mathbb{G}^{\mathrm{der}}$ and $\mathbb{G}/Z(\mathbb{G})$ and hence also their parabolic subgroups. For more details, see [Bor91, Theorem 22.6].

We now restrict to the F-points of our constructions above. So let $G = \mathbb{G}(F)$ and write $S = S_\varnothing$ and $P = P_\varnothing$ for the F-points of $\mathbb{S} = \mathbb{S}_\varnothing$ and $\mathbb{P} = \mathbb{P}_\varnothing$, respectively.

Definition 4.3. For $\Theta \subset \Delta$ and $\varepsilon \in (0, 1]$, define:

$$S_\Theta^-(\varepsilon) := \{s \in S_\Theta : |\alpha(s)| \leq \varepsilon \text{ for all } \alpha \in \Delta \setminus \Theta\}.$$

[Cas95, §1.4]

We simply write $S_\Theta^- = S_\Theta^-(1)$. If P is any parabolic subgroup of G, choose $g \in G$ so that $gPg^{-1} = P_\Theta$ for some $\Theta \subset \Delta$ and define $S^-(\varepsilon) := g^{-1}S_\Theta^-(\varepsilon)g$. Let $S^- := S^-(1)$.

Example 4.4. Let $\mathbb{G} = \mathrm{GL}_n / \mathbb{Q}_p$ and $G = \mathrm{GL}_n(\mathbb{Q}_p)$. We fix S to be the diagonal torus and P to be the Borel subgroup of upper triangular matrices. Then one has:

$$S^- = \{\mathrm{diag}(s_1, \ldots, s_n) \in \mathrm{GL}_n(\mathbb{Q}_p) : |s_1| \leq \cdots \leq |s_n|\}.$$

Let N be any locally compact group such that open compact subgroups form a basis of neighbourhoods about the identity, and possessing arbitrarily large open compact subgroups as well. This means that if X is a compact subset of N, then there is an open compact subgroup N_0 of N so that $X \subset N_0$. This condition is satisfied, for example, when N is the F-points of a unipotent group defined over F.

Definition 4.5. Let (π, V) be a smooth representation of N. For an open compact subgroup N_0 of N, define:

$$V(N_0) := \left\{v \in V : \int_{N_0} \pi(n)v \; dn = 0\right\}.$$

Definition 4.6. Let (π, V) be a smooth representation of N. Define:

$$V(N) := \bigcup_{N_0} V(N_0)$$

where N_0 runs over all open compact subgroups N_0 of N. By our assumption that N possesses arbitrarily large open compact subgroups, this is a subspace of V.

Proposition 4.7. $V(N) = \mathrm{span}\{\pi(n)v - v : n \in N, v \in V\}$.
[Cas95, Proposition 3.2.1]

Let P be any proper parabolic subgroup of G, and $P = MN$ be a Levi decomposition. Let (π, V) be a complex admissible representation of G. Define:

$$V_N := V/V(N).$$

The restricted action of P on V_N is trivial on N, so V_N admits a natural action of the Levi factor $M = P/N$. Let (π_N, V_N) denote this representation of M called the *Jacquet module* of (π, V). The (smooth) *Jacquet functor*

$$J_P : \{\text{smooth } G\text{-representations}\} \to \{\text{smooth } M\text{-representations}\}$$
$$(\pi, V) \mapsto (\pi_N, V_N)$$

is exact and admissible, but admissibility is non-trivial to show. It is a consequence of the following more precise statement.

Theorem 4.8. *Let (π, V) be an admissible representation of G, K_0 a compact open subgroup of G with Iwahori decomposition $K_0 = N_0^- M_0 N_0 = N_0 M_0 N_0^-$ with respect to P. Then the canonical projection $V^{K_0} \to V_N^{M_0}$ is surjective.*
[Cas95, Theorem 3.3.3]

This proves admissibility because there is a neighbourhood basis of open compact subgroups $\{K_n\}_{n\geq 0}$ of G each of which admits an Iwahori decomposition $K_n = N_n M_n N_n^-$ with respect to the Levi decomposition $P = MN$ [Cas95, Proposition 1.4.4]. Since $M \subset G$ has the subspace topology, $\{M_n = M \cap K_n\}_{n\geq 0}$ is a neighbourhood basis of M. This proves the admissibility of V_N from the admissibility of V.

Fix a parabolic subgroup P of G with $P = MN$. Let S denote the maximal F-split torus inside P. Let S^- be the subset of toral elements corresponding to these choices.

Definition 4.9. For a smooth representation (π, V) of G, and K open compact in G, define:

$$\mathcal{P}_K(v) := \frac{1}{\text{meas}(K)} \int_K \pi(k)v \; dk.$$

The smoothness of π ensures that this is a finite sum, and hence makes sense. The operator $\mathcal{P}_K$ is the projection of V onto V^K so that $V = V^K \oplus V(K)$.
[Cas95, p. 20]

Definition 4.10. For each open compact subgroup K of G. Let $\mathcal{H}(G, K)$ be the vector space of compactly supported K-biinvariant complex-valued functions on G. It has an algebra structure with respect to a natural convolution, and we call it the *Hecke algebra* of G with respect to (or relative to) K. Define the *Hecke algebra* of G to be $\mathcal{H}(G) := \bigcup_K \mathcal{H}(G, K)$ where the union is taken over all open compact subgroups K of G. It has an algebra structure induced from the algebra structures of $\mathcal{H}(G, K)$ for all K. If (π, V) is a smooth representation of G, then V becomes an $\mathcal{H}(G)$-module via the formula:

$$\pi(f)v := \int_G f(g)\pi(g)v \; dg.$$

This degenerates into a finite sum, because π is smooth. Moreover, there is an action of $\mathcal{H}(G, K)$ on V^K via the same formula, which is commonly used.

Definition 4.11. Let (π, V) be a smooth representation of G. Let K be an open compact subgroup of G. For any $g \in G$, let $[KgK]$ denote the indicator function of KgK in $\mathcal{H}(G, K)$ and let $[KgK]$ act on V (or V^K) via the Hecke algebra action.

Lemma 4.12. *Let (π, V) be a smooth representation of G. Let K_0 be an open compact subgroup of G admitting an Iwahori decomposition with respect to $P = MN$. Then for $s_1, s_2 \in S^-$, one has $[K_0 s_1 K_0][K_0 s_2 K_0] = [K_0 s_1 s_2 K_0]$. This turns*

$$\{[K_0 s K_0] : s \in S^-\}$$

into an abelian submonoid of $\mathcal{H}(G, K_0)$ isomorphic to S^-.
[Cas95, Lemma 4.1.5]

Lemma 4.13. *Let $g \in G$. Let K be a compact open subgroup of G. Then the natural map $K \to KgK$ sending $k \mapsto kg$ induces an isomorphism of right coset spaces*

$$K/(K \cap gKg^{-1}) \to KgK/K.$$

Lemma 4.14. *Let (π, V) be a smooth representation of G. Let K be an open compact subgroup of G. Then for $g \in G$ and $v \in V^K$, one has:*

$$\mathcal{P}_K(\pi(g)v) = \frac{\mathrm{meas}(K \cap gKg^{-1})}{\mathrm{meas}(K)^2}[KgK]v.$$

Proof. It is a simple calculation.

$$\begin{aligned}
\mathcal{P}_K(\pi(g)v) &= \frac{1}{\mathrm{meas}(K)} \int_K \pi(k)\pi(g)v \; dk \\
&= \frac{\mathrm{meas}(K \cap gKg^{-1})}{\mathrm{meas}(K)} \int_{K/(K\cap gKg^{-1})} \pi(k)\pi(g)v \; dk \\
&= \frac{\mathrm{meas}(K \cap gKg^{-1})}{\mathrm{meas}(K)} \int_{KgK/K} \pi(h)v \; dh && \text{(Lemma 4.13)} \\
&= \frac{\mathrm{meas}(K \cap gKg^{-1})}{\mathrm{meas}(K)^2} \int_{KgK} \pi(h)v \; dh \\
&= \frac{\mathrm{meas}(K \cap gKg^{-1})}{\mathrm{meas}(K)^2}[KgK]v.
\end{aligned}$$

□

Lemma 4.15. *Let $\delta_P : P \to \mathbb{C}^\times$ denote the modulus character of P. Let K_0 be an open compact subgroup of G admitting an Iwahori decomposition with respect to $P = MN$. Then for any $s \in S^-$, one has:*

$$[K_0 s K_0 : K_0] = [K_0 : K_0 \cap sK_0 s^{-1}] = \delta_P^{-1}(s).$$

[Cas95, Lemma 1.5.1]

Remark 4.16. Let (π, V) be a smooth representation of G. The significance of the previous two lemmas is that if we normalize the Haar measure on G so that $\mathrm{meas}(K_0) = 1$, then for $s \in S^-$ and $v \in V^{K_0}$, the integral formula:

$$s \cdot v := \mathcal{P}_{K_0}(\pi(s)v)$$

defines an action of the abelian monoid S^- on V.

Let (π, V) be a smooth admissible representation of G. Let K_0 be an open compact subgroup of G with Iwahori decomposition $K_0 = N_0 M_0 N_0^-$ with respect to $P = MN$. Assume the Haar measure on G is chosen so that $\mathrm{meas}(K_0) = 1$. For each $s \in S^-$, denote:

$$V_s^{K_0} := \mathcal{P}_{K_0}(\pi(s)V^{K_0}) = [K_0 s K_0]V^{K_0}.$$

Fix N_1 an open compact subgroup of N such that $V^{K_0} \cap V(N) \subset V(N_1)$. This is possible because V^{K_0} is finite-dimensional (by admissibility) and $V(N) = \bigcup_{N_0} V(N_0)$.

Proposition 4.17. *For $s \in S^-$, the natural projection $V_s^{K_0} \to V_N^{M_0}$ is a surjection. If $sN_1s^{-1} \subset N_0$, then $V_s^{K_0} \cap V(N) = 0$ so that the map is a bijection.*
[Cas95, Proposition 4.1.4]

Remark 4.18. There always exists some $s \in S^-$ such that $sN_1s^{-1} \subset N_0$. This is discussed in the paragraph just before [Eme06a, Proposition 4.3.4].

Proposition 4.19. *For $s \in S^-$ such that $sN_1s^{-1} \subset N_0$, the spaces $V_s^{K_0}$ are identical.*
[Cas95, Proposition 4.1.6]

In this case, denote this space by $V_{S^-}^{K_0}$. This is the *finite-slope* subspace of V^{K_0} for the operators $[K_0 s K_0]$ for $s \in S^-$. It is the maximal subspace of V^{K_0} for which one of, equivalently all of, these operators act invertibly. By the above result, there is an isomorphism $V_{S^-}^{K_0} \xrightarrow{\sim} V_N^{M_0}$. The inverse map $V_N^{M_0} \to V_{S^-}^{K_0}$ is called the *canonical lifting.*

Theorem 4.20. *Let $s \in S^-$ be any element. Then the action of $[K_0 s K_0]$ on $V_{S^-}^{K_0}$ is invertible.* [Cas95, Lemma 4.1.7]

The next result tells us that the monoidal action of S^- on V^{K_0} via Hecke operators agrees, up to a constant, with the toral action of M on the Jacquet module V_N.

Lemma 4.21. *If $v \in V^{K_0}$ has image $u \in V_N$, then for any $s \in S^-$, the image of $\mathcal{P}_{K_0}(\pi(s)v)$ under the Jacquet functor is equal to $\pi_N(s)u$.*
[Cas95, Lemma 4.1.1]

Remark 4.22. Recall it is "up to a constant" because the Hecke operators $[K_0 s K_0]v$ only agree with $\mathcal{P}_{K_0}(\pi(s)v)$ up to a constant. If we want the action of S^- on V^{K_0} to match up exactly with its action on V_N, then we ought to define it with $\mathcal{P}_{K_0}(\pi(s)v)$. This is exactly what we will do in the context of the soon-to-be-seen locally analytic Jacquet modules.

The next example should hopefully motivate our constructions.

Example 4.23. Let (π, V) be an admissible smooth representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let $P = MN$ be the Levi decomposition of the standard upper triangular Borel. Let

$$K_n = \begin{pmatrix} 1 + p^n\mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p^n\mathbb{Z}_p \end{pmatrix}$$

be a family of open compact neighbourhoods shrinking down to

$$N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}.$$

Each K_n has an Iwahori decomposition $K_n = N_0 M_n N_n^- = N_n^- M_n N_0$ where

$$M_n = \begin{pmatrix} 1 + p^n\mathbb{Z}_p & \\ & 1 + p^n\mathbb{Z}_p \end{pmatrix} \qquad \text{and} \qquad N_n^- = \begin{pmatrix} 1 & \\ p^n\mathbb{Z}_p & 1 \end{pmatrix}.$$

Moreover, N is the union of its open compact subgroups of the form:

$$N_n = \begin{pmatrix} 1 & p^n\mathbb{Z}_p \\ & 1 \end{pmatrix}.$$

In this case, S^- is the set of diagonal elements with non-decreasing valuations going down. The element $s_p := \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ is classically associated to the $U_p = [N_0 s_p N_0]$ operator. We wish to show that our above discussion produces:

(i) the finite-slope subspace $V_{\mathrm{fs}}^{N_0} \subset V^{N_0}$ corresponding to U_p, and
(ii) an isomorphism $V_{\mathrm{fs}}^{N_0} \xrightarrow{\sim} V_N$ with the smooth Jacquet module.

For each K_n, there exists $r \geq 1$ such that:

(i) $V^{K_n} \cap V(N) \subset V(N_{-r})$, and
(ii) $s_p^r N_{-r} s_p^{-r} \subset N_0$.

This lets us define $V_{S^-}^{K_n} := [K_n s_p^r K_n]V^{K_n}$. Let $V_{\mathrm{fs}}^{K_n} := \bigoplus_{\lambda\neq 0} E_\lambda$ be the direct sum of generalized λ-eigenspaces of $[K_n s_p K_n]$ for $\lambda \neq 0$. This is the maximal subspace of V^{K_n} on which $[K_n s_p K_n]$ acts invertibly. The general theory that we have developed above tells us that $[K_n s_p K_n]$ acts invertibly on $V_{S^-}^{K_n}$ so that in particular $V_{S^-}^{K_n} \subset V_{\mathrm{fs}}^{K_n}$. We will now show that $V_{S^-}^{K_n} = V_{\mathrm{fs}}^{K_n}$ so that our use of the term *finite-slope subspace* for $V_{S^-}^{K_n}$ is justified.

Lemma 4.24. $V_{S^-}^{K_n} = V_{\mathrm{fs}}^{K_n}$.

Proof. Let $v \in V_{\mathrm{fs}}^{K_n}$. Then setting J to be the inverse of $[K_n s_p K_n]$ on $V_{\mathrm{fs}}^{K_n}$ one gets:
$$v = [K_n s_p K_n]^r (J^r v) = [K_n s_p^r K_n](J^r v) \in V_{S^-}^{K_n}.$$ □

Suppose now $v \in V^{N_0}$. Since V is a smooth representation of G, the stabilizer of v is open and contains N_0, and hence contains an open compact subgroup K_n for some n, because:
$$N_0 = \bigcap_{n\geq 1} K_n.$$
This is equivalent to the statement that:
$$\bigcup_{n\geq 1} V^{K_n} = V^{N_0}.$$
Let $v \in V^{N_0}$, then $v \in V^{K_n}$ for some n. By Jacquet's First Lemma [Cas95, Theorem 3.3.4], the action of $[N_0 s_p N_0]$ on $v \in V^{N_0}$ agrees with the action of $[K_n s_p K_n]$ on $v \in V^{K_n}$, albeit only up to a constant. Nevertheless, this implies that $U_p = [N_0 s_p N_0]$ stabilizes V^{K_n}, and in particular U_p is a *locally finite* operator. Let $V_{\mathrm{fs}}^{N_0}$ denote the maximal subspace of V^{N_0} on which U_p acts invertibly. This is the same as the subspace of vectors $v \in V^{N_0}$ such that U_p becomes invertible when restricted to the subspace:
$$\mathrm{span}\{v, U_p v, U_p^2 v, \dots\}.$$
Then it is clear by our discussion that:
$$V_{\mathrm{fs}}^{N_0} = \bigcup_{n\geq 1} V_{S^-}^{K_n} = \bigcup_{n\geq 1} V_{\mathrm{fs}}^{K_n}.$$
Finally, we obtain a natural map:
$$V_{\mathrm{fs}}^{N_0} = \bigcup_{n\geq 1} V_{S^-}^{K_n} \to \bigcup_{n\geq 1} V_N^{M_n} = V_N.$$
It is clearly surjective. Moreover, the restriction of this map to each $V_{S^-}^{K_n}$ is an isomorphism, implying that this map is injective. Therefore, it is an isomorphism. This is the connection, as we shall see, between the smooth Jacquet module, and the somewhat different definition involved in Emerton's locally analytic Jacquet module.

We now want to define a locally analytic version of the Jacquet functor. Before we do this, let us just make a quick remark about the finite-slope subspace, which we have defined to be the maximal subspace on which some operator (typically U_p) is invertible.

Let V be a finite-dimensional complex vector space equipped with the action of a linear operator T, so that in particular V is a $\mathbb{C}[T]$-module. We can write:
$$V_{\mathrm{fs}} := \mathrm{Hom}_{\mathbb{C}[T]}(\mathbb{C}[T, T^{-1}], V).$$
Then V_{fs} has a $\mathbb{C}[T, T^{-1}]$-module structure via (left or right) translation on the domain.

Remark 4.25. For $\Phi \in V_{\text{fs}}$, the image of Φ is a T-invariant subspace of V on which T acts invertibly, and this subspace is spanned by the vectors $\{T^k\Phi(1) : k \in \mathbb{Z}\}$.

For $\Phi \in V_{\text{fs}}$, evaluating at the identity, that is $\Phi \mapsto \Phi(1)$, induces a natural $\mathbb{C}[T]$-module homomorphism $V_{\text{fs}} \to V$ that is an isomorphism onto its image:

$$V_{\text{fs}} \xrightarrow{\sim} \bigoplus_{\lambda \neq 0} E_\lambda \subset V.$$

Here E_λ denotes the generalized eigenspace of λ where λ is a root of the characteristic polynomial of T. This direct sum is the maximal subspace on which T acts invertibly, and although we had previously denoted this space "V_{fs}", this isomorphism hopefully justifies our current abuse of notation. Since T acts invertibly on both sides, the $\mathbb{C}[T]$-module isomorphism upgrades to a $\mathbb{C}[T, T^{-1}]$-module isomorphism for free.

Some categorical properties of this construction include [Eme06a, §3.2]:

(a) The functor $\mathrm{Hom}_{\mathbb{C}[T]}(\mathbb{C}[T, T^{-1}], -)$ is right adjoint to the forgetful functor from the category of $\mathbb{C}[T, T^{-1}]$-modules to the category of $\mathbb{C}[T]$-modules.
(b) The natural structure map $V_{\text{fs}} \to V$ realizes V_{fs} as the final object in the category of $\mathbb{C}[T, T^{-1}]$-modules equipped with a $\mathbb{C}[T]$-linear map to V.

We use this perspective of finite slope to define Emerton's locally analytic Jacquet functor. To set things up, we need a few more definitions from locally analytic representation theory. Let F be a finite extension of $\mathbb{Q}_p$, and E be an extension of F, complete with respect to a discrete valuation extending the discrete valuation of F.

Definition 4.26. If G is a topological group (or semigroup), let $\mathrm{Rep}_{\text{top.c}}(G)$ denote the category of Hausdorff locally convex E-vector spaces of compact type, equipped with a topological action of G, and whose morphisms are continuous G-equivariant E-linear maps. [Eme06a, §3.1]

Definition 4.27. If G is a locally F-analytic group, let $\mathrm{Rep}_{\text{la.c}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\text{top.c}}(G)$ consisting of locally analytic representations of G on locally convex E-vector spaces of compact type.
[Eme06a, §3.1]

Definition 4.28. Let G be a locally F-analytic group, and suppose the centre Z_G of G is topologically finitely generated. Let $\mathrm{Rep}_{\text{es}}(G)$ be the full subcategory of $\mathrm{Rep}_{\text{la.c}}(G)$ consisting of essentially admissible locally analytic representations of G. Let $\mathrm{Rep}_{\text{ad}}(G)$ be the full subcategory of $\mathrm{Rep}_{\text{es}}(G)$ consisting of admissible locally analytic representations of G.
[Eme06a, §3.1]

Definition 4.29. Let G be a locally F-analytic group. Let $\mathrm{Rep}^{z}_{\text{la.c}}(G)$ be the full subcategory of $\mathrm{Rep}_{\text{la.c}}(G)$ consisting of locally convex E-vector spaces V of compact type, equipped with a locally analytic representation of G, and such that V may be written as the union of an increasing sequence of Z_G-invariant BH-subspaces.
[Eme06a, §3.1]

This last condition about V being written as the union of some increasing sequence of Z_G-invariant BH-subspaces can be reinterpreted as follows.

Proposition 4.30. *Let V be a locally convex E-vector space of compact type, equipped with a topological action of the topologically finitely generated abelian locally F-analytic group Z. The following conditions are equivalent:*

(i) The Z-action on V extends to a $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$-module structure for which the multiplication map $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E) \times V \to V$ is separately continuous.
(ii) The contragredient Z-action on V'_b extends to a topological $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$.
(iii) The Z-action on V is locally analytic, and we may write V as a union of an increasing sequence of BH-subspaces, each invariant under Z.

[Eme17, Proposition 6.4.7]

Fix a topologically finitely-generated abelian locally F-analytic group Z. We also fix a topological submonoid Z^+ of Z such that Z^+ generates Z as a group, and contains a compact open subgroup of Z. There is an obvious forgetful functor $\mathrm{Rep}^z_{\mathrm{la.c}}(Z) \to \mathrm{Rep}_{\mathrm{top.c}}(Z^+)$. We will define a right adjoint to this functor by taking "finite slope".

Definition 4.31. For any object V of $\mathrm{Rep}_{\mathrm{top.c}}(Z^+)$, we write:

$$V_{\mathrm{fs}} := \mathcal{L}_{b,Z^+}(\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E), V).$$

($\mathcal{L}_{b,Z^+}(\dots)$ is the subspace of $\mathcal{L}_b(\dots)$ consisting of Z^+-equivariant maps.)
[Eme06a, Definition 3.2.1]

Remark 4.32. Recall that if V and W are two locally convex E-vector spaces, then $\mathcal{L}(V, W)$ denotes the E-vector space of continuous E-linear maps from V to W, and $\mathcal{L}_b(V, W)$ denotes the underlying space $\mathcal{L}(V, W)$ equipped with its strong topology; see [ST04, §7] for details.

Remark 4.33. Evaluation at a point induces a natural inclusion of Z into the group of units of $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$. This gives a Z-module structure on $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$, and hence also V_{fs} via translation in the domain. (One needs to check that this sends $\mathcal{L}_{b,Z^+}(\dots)$ into itself.)

The space $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$ is our replacement in the locally analytic setting for the perhaps more intuitive object $E[Z]$. There is a natural map of E-algebras $E[Z] \to \mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$ induced by $z \mapsto \delta_z$ where δ_z is the evaluation at z map. This embedding has dense image; this is mentioned in the proof of [Eme17, Lemma 6.4.8].

Example 4.34. Let $Z^+ = \mathbb{N}$ and $Z = \mathbb{Z}$. Since $\mathbb{Z}$ has the discrete topology, it is in fact a locally F-analytic group, of dimension zero! Since any character of Z is determined entirely by the image of $1 \in \mathbb{Z}$, one has $\widehat{Z} = \mathbb{G}_m/F$ in the notation of [Eme17, Proposition 6.4.5]. Let $\widehat{Z}_E$ denote the base change of $\widehat{Z}$ to E. Then

$$\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E) := \mathcal{O}_{\widehat{Z}_E}(\widehat{Z}_E) \xrightarrow{\sim} E\{\!\{x, x^{-1}\}\!\}.$$

Here the element x corresponds to the natural inclusion $\mathbb{G}_{m,E} \hookrightarrow \mathbb{A}^1_E$ in $\mathrm{Hom}(\mathbb{G}_{m,E}, \mathbb{A}^1_E)$ and can be interpreted as the element in $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$ which sends any character in $\widehat{Z}$ to its evaluation at $1 \in \mathbb{Z}$. Therefore, the map $E[Z] \to \mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$ has dense image $E[x, x^{-1}]$. Similarly, the map $E[Z^+] \to \mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$ has image $E[x]$.

Remark 4.35. In [Eme17, Definition 2.1.9], he defines $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, E)$ to be the E-Banach algebra of E-valued rigid analytic functions defined on $\mathbb{X}$, where $\mathbb{X}$ is any affinoid rigid analytic space defined over F. He does not mean the space of functions "$\mathbb{X} \to E$", since the domain and codomain are not even in the same category. He probably means for $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, E)$ to denote $\mathrm{Hom}(\mathbb{X}_E, \mathbb{A}^1_E) = \mathcal{O}(\mathbb{X}_E)$ where $\mathbb{X}_E = \mathbb{X} \otimes_F E$, and $\mathbb{A}^1_E$ is the rigid affine line over E. This would make sense because $\mathcal{O}(\mathbb{X}_E)$ is indeed an E-Banach space.

The following lemma provides an easy description for the strong dual of V_{fs}.

Lemma 4.36. *If V is an object of $\mathrm{Rep}_{\mathrm{top.c}}(Z^+)$, then there is a natural isomorphism:*

$$(V_{\mathrm{fs}})'_b = \mathcal{C}^{\mathrm{an}}(\widehat{Z}, E) \ \widehat{\otimes}_{E[Z^+]} \ V'_b.$$

[Eme06a, Lemma 3.2.3]

We are almost ready to define the locally analytic Jacquet functor. Before doing so, we need to introduce an analogue of the Hecke algebra action in this setting. Fix a connected reductive group $\mathbb{G}$ defined over F, a parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$, and write $\mathbb{P} = \mathbb{M}\mathbb{N}$ for its Levi decomposition. Write $G := \mathbb{G}(F)$, $P := \mathbb{P}(F)$, $M := \mathbb{M}(F)$, and $N := \mathbb{N}(F)$. Let $\mathbb{Z}_{\mathbb{G}}$ (respectively $\mathbb{Z}_{\mathbb{M}}$) denote the centre of $\mathbb{G}$ (respectively $\mathbb{M}$), and write $Z_G := \mathbb{Z}_{\mathbb{G}}(F)$ (respectively $Z_M := \mathbb{Z}_{\mathbb{M}}(F)$), then Z_G (respectively Z_M) is the centre of G (respectively M).

Let $\mathfrak{n}$ denote the Lie algebra of N. Let V be a locally analytic representation of P, and let $V^{\mathfrak{n}}$ denote the closed subspace of $\mathfrak{n}$-invariants of V. The subspace $V^{\mathfrak{n}}$ is P-invariant, since $\mathfrak{n}$ is invariant under the adjoint action of P. Note that when regarded as a locally analytic representation of N, the subspace $V^{\mathfrak{n}}$ is precisely the subspace of smooth vectors of V, since recall that $(-)_{\mathrm{sm}} = (-)^{\mathfrak{n}}$, and so there is a smooth N-action on $V^{\mathfrak{n}}$.

Definition 4.37. Let (π, V) be a locally analytic representation of P. If N_0 is any open compact subgroup of N, define the projection operator $\mathcal{P}_{N_0} : V^{\mathfrak{n}} \to V^{N_0}$ as follows:

$$\mathcal{P}_{N_0}(v) := \int_{N_0} \pi(n)v \ dn.$$

(The measure dn is the Haar measure on N, normalized so that N_0 has measure one.)
[Eme06a, Definition 3.4.1]

Fix a compact open subgroup P_0 of P, and write $M_0 := M \cap P_0$, $N_0 := N \cap P_0$, $M^+ := \{m \in M : mN_0m^{-1} \subset N_0\}$, and $Z_M^+ := M^+ \cap Z_M$. Note $M_0 \subset M^+$ and $Z_G \subset Z_M^+$.

Proposition 4.38. *The abelian group Z_M is generated by its submonoid Z_M^+.*
[Eme06a, Corollary 3.3.3]

Example 4.39. Let $\mathbb{G} := \mathrm{GL}_2 / \mathbb{Q}_p$. Let $\mathbb{P} = \mathbb{M}\mathbb{N}$ be the Levi decomposition of the Borel upper triangular matrices in $\mathbb{G}$ into the diagonal torus and its unipotent radical. Let

$$P_0 := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ & \mathbb{Z}_p^\times \end{pmatrix}$$

be a compact open subgroup of $P := \mathbb{P}(\mathbb{Q}_p)$. Recall $M = \begin{pmatrix} \mathbb{Q}_p^\times & \\ & \mathbb{Q}_p^\times \end{pmatrix}$. Then

$$\begin{aligned} M_0 &:= \begin{pmatrix} \mathbb{Z}_p^\times & \\ & \mathbb{Z}_p^\times \end{pmatrix} \\ N_0 &:= \begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix} \\ M^+ &:= \{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p) : |t_1| \leq |t_2| \} \end{aligned}$$

So M^+ is *exactly* the monoid S^- defined earlier!

Definition 4.40. If $m \in M^+$, define $\mathcal{P}_{N_0,m} : V^{\mathfrak{n}} \to V^{N_0}$ as follows:

$$\mathcal{P}_{N_0,m}(v) := \mathcal{P}_{N_0}(mv).$$

[Eme06a, Definition 3.4.2]

For $m \in M^+$, note that the restriction of $\mathcal{P}_{N_0,m}$ to V^{N_0} induces an endomorphism of V^{N_0}.

Lemma 4.41. *If $m \in M^+$, then the endomorphism $\mathcal{P}_{N_0,m}$ of V^{N_0} is continuous.*
[Eme06a, Lemma 3.4.3]

Lemma 4.42. *If $m, m' \in M^+$, then $\mathcal{P}_{N_0,m}\mathcal{P}_{N_0,m'} = \mathcal{P}_{N_0,mm'}$.*
[Eme06a, Lemma 3.4.4]

So the operators $\mathcal{P}_{N_0,m}$ define a topological action of the monoid M^+ on V^{N_0}. In particular, we obtain a topological action of Z_M^+ on V^{N_0}. Suppose now that V is of compact type, so that V is an object in the category $\mathrm{Rep}_{\mathrm{la.c}}(P)$. Then the closed subspace V^{N_0} of V is also of compact type, and is equipped with a topological action of Z_M^+. Thus it is an object in the category $\mathrm{Rep}_{\mathrm{top.c}}(Z_M^+)$. We can take its finite slope part.

Definition 4.43. If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$, we can view V^{N_0} as an object of $\mathrm{Rep}_{\mathrm{top.c}}(Z_M^+)$, and then define $J_P(V) := (V^{N_0})_{\mathrm{fs}}$.
[Eme06a, Definition 3.4.5]

By [Eme06a, Proposition 3.2.4], this functor outputs an object in $\mathrm{Rep}^z_{\mathrm{la.c}}(Z_M)$.

Proposition 4.44. *If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$, then the locally analytic Z_M-representation on $J_P(V)$ extends in a natural way to a locally analytic M-representation.*
[Eme06a, Proposition 3.4.6]

Thus J_P defines a functor $\mathrm{Rep}_{\mathrm{la.c}}(P) \to \mathrm{Rep}^z_{\mathrm{la.c}}(M)$. In [Eme06a, §3.5], this will be classified as an adjoint functor to some functor. Since there is a natural forgetful functor from $\mathrm{Rep}_{\mathrm{la.c}}(G)$ to $\mathrm{Rep}_{\mathrm{la.c}}(P)$ by restricting the group action, we also obtain a functor:

$$J_P : \mathrm{Rep}_{\mathrm{la.c}}(G) \to \mathrm{Rep}^z_{\mathrm{la.c}}(M).$$

Recall that $\mathrm{Rep}_{es}(G)$ is a full subcategory of $\mathrm{Rep}_{\mathrm{la.c}}(G)$.

Theorem 4.45. *If V is an object of $\mathrm{Rep}_{\mathrm{es}}(G)$, then the Jacquet module $J_P(V)$ is an object of $\mathrm{Rep}_{\mathrm{es}}(M)$. Thus J_P induces a functor:*

$$J_P : \mathrm{Rep}_{\mathrm{es}}(G) \to \mathrm{Rep}_{\mathrm{es}}(M).$$

[Eme06a, Theorem 4.2.32]

Let V be an admissible smooth representation of G defined over E. If we equip V with its finest convex topology, then it becomes an object of $\mathrm{Rep}_{\mathrm{ad}}(G)$. One can show that $J_P(V)$ is isomorphic as an M-representation to the N-coinvariants V_N of V. Indeed, this is just saying that the locally analytic Jacquet functor J_P, when restricted to admissible smooth representations, recovers the classical smooth Jacquet functor.

Proposition 4.46. *(a) The natural quotient map $V \to V_N$ induces an M^+-equivariant surjective map $V^{N_0} \to V_N$ (the M^+-action is defined on the source via operators $\mathcal{P}_{N_0,m}$, and on the target via restricting the natural M-action).*
(b) The M^+-equivariant map $V^{N_0} \to V_N$, after passing to the finite slope part on the source, restricts to an M-equivariant isomorphism $J_P(V) \xrightarrow{\sim} V_N$.
[Eme06a, Proposition 4.3.4]

There is an easy description of J_P when applied to locally algebraic representations.

Proposition 4.47. *Let W be a finite-dimensional algebraic representation of $\mathbb{G}$, and let $B = \mathrm{End}_G(W)$. If U is an admissible smooth representation of G over B, then there is a natural M-equivariant isomorphism:*

$$J_P(U \otimes_B W) \xrightarrow{\sim} U_N \otimes_B W^N.$$

[Eme06a, Proposition 4.3.6]

Recall that a locally analytic representation V of G is said to be an admissible locally algebraic representation of G if it, when endowed with its finest convex topology, becomes an admissible locally analytic representation of G. We will always regard an admissible locally algebraic representation of G as being an object of $\mathrm{Rep}_{\mathrm{ad}}(G)$, by endowing it with its finest convex topology. By [Eme17, Proposition 6.3.11], any admissible locally algebraic representation V of G admits an isomorphism:

$$V \xrightarrow{\sim} \bigoplus_n U_n \otimes_{B_n} W_n$$

where W_n runs over a sequence of isomorphism class representatives for the irreducible algebraic finite-dimensional representations of $\mathbb{G}$, $B_n := \mathrm{End}_G(W_n)$, and U_n is an admissible smooth representation of G over B_n. Then the additivity of J_P [Eme06a, Lemma 3.4.7] together with Proposition 4.47 tell us that:

$$J_P(V) \xrightarrow{\sim} \bigoplus_n (U_n)_N \otimes_{B_n} W_n^N.$$

The final property of the locally analytic Jacquet functor J_P that we want to discuss in this section is: when does it commute with the functor of taking locally algebraic vectors? As we shall see in the next section, the answer to this question gives rise to a version of "small slope implies classical" in the locally analytic setting. The idea behind what we want to do is as follows. Let V be an essentially admissible locally analytic representation of G. Let W be an irreducible finite-dimensional algebraic representation of G. The natural closed embedding $V_{W\text{-lalg}} \to V$ induces a closed embedding of M-representations:

$$J_P(V_{W\text{-lalg}}) \to J_P(V).$$

If χ is a character of Z_M, then this restricts to an embedding on χ-eigenspaces:

$$(V_{W\text{-lalg}})^{N_0, Z_M^+ = \chi} = (V_{W\text{-lalg}})_{\mathrm{fs}}^{N_0, Z_M = \chi} =: J_P(V_{W\text{-lalg}})^{\chi} \to J_P(V)^{\chi}.$$

The left hand side is the space of classical (locally W-algebraic) eigenforms for which the U_p-operator (the monoid Z_M^+) acts via a finite-slope eigenvalue (an invertible character χ). The right hand side is the *a priori* bigger space of p-adic (locally analytic) eigenforms, which interpolate between the classical eigenforms, and for which the U_p-operator (the group Z_M) acts by the same eigenvalue (character). If we are given a p-adic eigenform, and are told that U_p acts on it via some finite-slope eigenvalue, can we deduce whether it is classical, by which we mean it lies in the image of the above embedding?

To lie in the classical subspace, there is an easy necessary condition that must be satisfied. Observe that the above embedding factors through an algebraic subspace in the codomain:

$$J_P(V_{W\text{-lalg}})^{\chi} \to J_P(V)^{\chi}_{W^N\text{-lalg}}.$$

Let ψ denote the character through which Z_M acts on W^N. (Such a ψ exists if we assume G to be split over the field E over which W is defined.) Note that ψ is basically the highest weight of W. If χ and ψ do not coincide locally, and so in particular if χ is not locally algebraic, then both the domain and codomain vanish.

There is also a sufficient condition. Suppose χ and ψ coincide locally. If V admits a G-invariant norm, and if χ is of "non-critical slope" (to be defined) with respect to the irreducible representation W^N, then the previous map is an isomorphism. (For this, we will need to assume that $F = \mathbb{Q}_p$, so that G is locally $\mathbb{Q}_p$-analytic, and that G splits over E.)

To this end, let us be more precise about what we mean. Let χ be an E-valued locally F-analytic character of Z_M, that is $\chi \in \widehat{Z}_M(E)$, and V be an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$. Write:

$$V^{N_0, Z_M^+ = \chi} := \{v \in V^{N_0} : \pi_{N_0,z} v = \chi(z) v \text{ for all } z \in Z_M^+\}.$$

We regard this as a locally analytic representation of Z_M, by having Z_M act via χ. Recall we can view V^{N_0} as an object in $\mathrm{Rep}_{\mathrm{top.c}}(Z_M^+)$ so that V^{N_0}, and hence its χ-eigenspace as a closed subspace, is of compact type as well. Furthermore, since Z_M acts on this χ-eigenspace through a scalar, we have certainly defined an object in $\mathrm{Rep}^z_{\mathrm{la.c}}(Z_M)$. On the other hand, let $J_P(V)^\chi$ denote the closed subrepresentation of $J_P(V)$ on which Z_M acts via χ.

Proposition 4.48. *If V is an object of $\mathrm{Rep}_{\mathrm{la.c}}(P)$, then the natural composition of maps $J_P(V) \to V^{N_0} \to V$ induces an isomorphism of objects in $\mathrm{Rep}^z_{\mathrm{la.c}}(Z_M)$:*

$$J_P(V)^\chi \xrightarrow{\sim} V^{N_0, Z_M^+ = \chi}.$$

[Eme06a, Proposition 3.4.9]

From this point onward, let $F = \mathbb{Q}_p$ so that $G := \mathbb{G}(\mathbb{Q}_p)$ is a locally $\mathbb{Q}_p$-analytic group, where $\mathbb{G}$ is a connected reductive group over $\mathbb{Q}_p$, and suppose $\mathbb{G}$ splits over E. Keep all other notation the same as for when we defined the J_P functor.

Let $\mathbb{S}$ denote the maximal subtorus of $\mathbb{Z}_{\mathbb{M}}$ that splits over $\mathbb{Q}_p$. Let $X^\bullet$ (respectively $X_\bullet$) denote the character (respectively cocharacter) lattice of $\mathbb{Z}_{\mathbb{M}}$ over $\overline{\mathbb{Q}}_p$. Similarly, let $Y^\bullet$ (respectively $Y_\bullet$) denote the character (respectively cocharacter) lattice of $\mathbb{S}$ (over $\mathbb{Q}_p$ or $\overline{\mathbb{Q}}_p$). There is a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\mathbb{Z}_{\mathbb{M}}$ and $\mathbb{G}_m$, and this extends to an action on $X^\bullet$ and $X_\bullet$ by conjugation, compatible with respect to the canonical pairing between them. The natural embedding $Y_\bullet \to X_\bullet$ identifies $Y_\bullet$ with the sublattice of Galois invariants in $X_\bullet$.

Let $S := \mathbb{S}(\mathbb{Q}_p)$ and $Z_M := \mathbb{Z}_{\mathbb{M}}(\mathbb{Q}_p)$. Let S^0 (respectively Z_M^0) denote the maximal compact subgroup of S (respectively Z_M). There is a natural map:

$$\begin{aligned} Z_M &\to \mathrm{Hom}(X^\bullet, \mathbb{Q}) = \mathrm{Hom}(X^\bullet, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = X_\bullet \otimes_{\mathbb{Z}} \mathbb{Q} \\ z &\mapsto \Big[\chi \mapsto \nabla_z(\chi) := \mathrm{ord}_{\overline{\mathbb{Q}}_p}(\chi(z))\Big]. \end{aligned}$$

The functions ∇_z are Galois invariant, so this map factors as:

$$Z_M \to Y_\bullet \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let $Y'_\bullet$ denote the image of this map. Since S is split, when restricted to S, there is a surjection $S \to Y_\bullet$, whose kernel is S^0. There is a diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^0 & \longrightarrow & S & \longrightarrow & Y_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_M^0 & \longrightarrow & Z_M & \longrightarrow & Y'_\bullet & \longrightarrow & 0. \end{array}$$

Since $Y_\bullet$ and $Y'_\bullet$ are both free $\mathbb{Z}$-modules of finite rank, each row is split.

Definition 4.49. Let $\widehat{Z}_M$ denote the rigid space of locally $\mathbb{Q}_p$-analytic characters of Z_M.

(i) Given $\chi \in \widehat{Z}_M(E)$, define a function $\Delta_\chi : Z_M \to \mathbb{Q}$ by the formula:

$$\Delta_\chi(z) := \mathrm{ord}_E(\chi(z)).$$

This is trivial on Z_M^0, and factors through a map $Y'_\bullet \to \mathbb{Q}$.

(ii) This yields a map $Y_\bullet \to Y'_\bullet \to \mathbb{Q}$, and hence an element of $Y^\bullet \otimes_\mathbb{Z} \mathbb{Q}$ via the isomorphism:

$$Y^\bullet \otimes_\mathbb{Z} \mathbb{Q} = \mathrm{Hom}(Y_\bullet, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q} = \mathrm{Hom}(Y_\bullet, \mathbb{Q}).$$

The element of $Y^\bullet \otimes_\mathbb{Z} \mathbb{Q}$ associated to χ is called the *slope* of χ, denoted $\mathrm{slope}(\chi)$. [Eme06a, Definition 1.4.2]

Example 4.50. Let $\mathbb{G} = \mathrm{GL}_2/\mathbb{Q}_p$ and $\mathbb{P}$ denote the standard Borel subgroup of upper triangular matrices. In this case, $\mathbb{S} = \mathbb{Z}_\mathbb{M}$ so the two rows of short exact sequences are equal. Let $\mathbb{T} := \mathbb{S} = \mathbb{Z}_\mathbb{M} = \mathbb{M}$ be the diagonal torus, and $T := \mathbb{T}(\mathbb{Q}_p)$. Then

$$X^\bullet = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m) = \{[a,b] \in \mathbb{Z}^2 : \mathrm{diag}(t_1,t_2) \mapsto t_1^a t_2^b\} \cong \mathbb{Z}^2$$
$$X_\bullet = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) = \{[e,f] \in \mathbb{Z}^2 : t \mapsto \mathrm{diag}(t^e, t^f)\} \cong \mathbb{Z}^2.$$

Here, $T^0 = \mathrm{diag}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$. Recall that the map $t \mapsto \nabla_t$ induces a natural isomorphism:

$$T/T^0 \xrightarrow{\sim} X_\bullet$$
$$\mathrm{diag}(p^e, p^f) \mapsto [e,f].$$

Let $\chi \in \widehat{T}(E)$ be an *unramified* character; this means $\chi : T \to E^\times$ satisfies that $\chi(T^0) = 1$. Let $\alpha := \chi\begin{pmatrix} p & \\ & 1\end{pmatrix}$ and $\beta := \chi\begin{pmatrix} 1 & \\ & p\end{pmatrix}$. Then χ is completely determined by the pair (α, β). We will show that $\mathrm{slope}(\chi) = [\mathrm{ord}_E(\alpha), \mathrm{ord}_E(\beta)]$ as an element in $X^\bullet \otimes_\mathbb{Z} \mathbb{Q}$. Indeed, being unramified, χ induces a character $T/T^0 \to E^\times$, and hence also:

$$\begin{array}{ccc} X_\bullet \to E^\times & & X_\bullet \to \mathbb{Q} \\ {[e,f] \mapsto \alpha^e\beta^f} & \xRightarrow{\mathrm{ord}_E} & {[e,f] \mapsto e\,\mathrm{ord}_E(\alpha) + f\,\mathrm{ord}_E(\beta).} \end{array}$$

This corresponds to the element $[\mathrm{ord}_E(\alpha), \mathrm{ord}_E(\beta)]$ in $X^\bullet \otimes_\mathbb{Z} \mathbb{Q}$, which is $\mathrm{slope}(\chi)$.

Let $\Delta(\mathbb{G}, \mathbb{S})$ denote the set of positive restricted roots of $\mathbb{S}$; these are the characters of $\mathbb{S}$ appearing in the adjoint action of $\mathbb{S}$ on the Lie algebra of $\mathbb{N}$. Let $\Delta(\mathbb{G}, \mathbb{S})_s$ denote the subset of positive simple restricted roots. Let $R^\bullet$ denote the sublattice of $Y^\bullet$ spanned by $\Delta(\mathbb{G}, \mathbb{S})_s$, and let $(R^\bullet \otimes_\mathbb{Z} \mathbb{Q})^{\geq 0}$ denote the cone in $R^\bullet \otimes_\mathbb{Z} \mathbb{Q}$ generated by the $\mathbb{Q}_{\geq 0}$-span of $\Delta(\mathbb{G}, \mathbb{S})_s$.

Let δ_P denote the modulus character of P, regarded as a smooth character of M. If we write $\rho := \rho(\mathbb{G}, \mathbb{S})$ as the one-half sum of the elements in $\Delta(\mathbb{G}, \mathbb{S})$, each counted with the multiplicity with respect to which it appears in the adjoint action of $\mathbb{S}$ on $\mathbb{N}$, then $\mathrm{slope}(\delta|_\mathbb{S}) = -\rho$. (This is where the assumption that $F = \mathbb{Q}_p$ is used.)

Let W be a finite-dimensional irreducible algebraic representation of $\mathbb{G}$ over E. Since $\mathbb{G}$ splits over E, the theory of highest weight tells us that the space $W^\mathbb{N}$ of $\mathbb{N}$-invariants of W is an irreducible algebraic representation of $\mathbb{M}$, and so $\mathbb{Z}_\mathbb{M}$ must act on $W^\mathbb{N}$ via a central character $\psi \in X^\bullet(\mathbb{Z}_\mathbb{M})$. We can regard ψ as being an element of $\widehat{Z}_M(E)$. We also fix a smooth (i.e. locally constant) character $\theta \in \widehat{Z}_M(E)$, and write $\chi := \theta\psi$.

Let us fix a Borel subgroup $\mathbb{B}$ of $\mathbb{G}$, defined over E. Let $\mathbb{T}$ be a maximal torus of this Borel subgroup, again defined over E, that is contained in $\mathbb{M}$. The intersection $\mathbb{M} \cap \mathbb{B}$ is then a Borel subgroup of $\mathbb{M}$. Let $\widetilde{\psi} \in X^\bullet(\mathbb{T})$ denote the highest weight of the representation W

with respect to $\mathbb{B}$; this is also the highest weight of the $\mathbb{M}$-representation $W^{\mathbb{N}}$ with respect to $\mathbb{M}\cap\mathbb{B}$. Note $\widetilde{\psi}|_{\mathbb{Z}_{\mathbb{M}}} = \psi$. Recall $\mathfrak{n}$ is the Lie algebra of N. Let $\mathfrak{n}'$ denote the Lie algebra of the unipotent radical of $\mathbb{B}$. If α is a simple restricted root of $\mathbb{Z}_{\mathbb{M}}$ acting on $\mathfrak{n}$, then denote by $\widetilde{\alpha}$ the simple root of $\mathbb{T}$ acting on $\mathfrak{n}'$ that lifts α. If $\widetilde{\rho} := \rho(\mathbb{G},\mathbb{T})$ is the one half sum of positive roots of $\mathbb{T}$ acting on $\mathfrak{n}'$, then $\widetilde{\rho}|_{\mathbb{S}} = \rho$.

Definition 4.51. We say that $\chi = \psi\theta$ is of *critical slope* with respect to the representation $W^{\mathbb{N}}$ (of $\mathbb{M}$) if for some positive simple root $\alpha \in \Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})$, the element

$$s_{\widetilde{\alpha}}(\widetilde{\psi} + \widetilde{\rho})|_{\mathbb{S}} + \mathrm{slope}(\theta) + \rho$$

of $Y^{\bullet} \otimes_{\mathbb{Z}} \mathbb{Q}$ in fact lies in the positive cone $(R^{\bullet} \otimes_{\mathbb{Z}} \mathbb{Q})^{\geq 0}$. Otherwise, we say that χ is of *non-critical slope* with respect to $W^{\mathbb{N}}$.
[Eme06a, Definition 4.4.3]

These are literally just random words until we do the following illuminating calculation.

Example 4.52. Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$, and $B = TN$ be the Levi decomposition of the standard upper triangular Borel subgroup of G. Let δ_B denote the modulus character of B. For $k \geq 2$, let $W_k := \mathrm{Sym}^{k-2}\,\mathbb{Q}^2$ be the irreducible representation of $\mathrm{GL}_2/\mathbb{Q}$, with highest weight ψ_k. Let $\Gamma := \mathrm{GL}_2(\mathbb{A}^{\infty,p})$ be the auxiliary locally compact group. Let $\mathbb{G} := \mathrm{GL}_2/\mathbb{Q}$.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2/\mathbb{Q}$ such that π_∞ is the discrete series representation of weight k (in the arithmetic normalization), and π_p is unramified; that is, the $\mathrm{GL}_2(\mathbb{Z}_p)$-fixed vectors in π_p form a one-dimensional subspace, or equivalently there exist unramified characters $\theta_1, \theta_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ such that if $\theta := \theta_1 \otimes \theta_2 : T \to \mathbb{C}^\times$ then:

$$\pi_p = \mathrm{Ind}_B^G\,\theta.$$

(This is the normalized induction.) Therefore, associated to π is some newform $f \in S_k(\Gamma_1(N))$ for some $p \nmid N$, such that $T_p f = a_p(f) f$. The functions $f(z)$ and $f(pz)$ span a two-dimensional subspace of oldforms in $\mathcal{S}_k(\Gamma_1(N) \cap \Gamma_0(p))$, which is preserved by the Atkin-Lehner U_p-operator at this level. The restriction of U_p to this two-dimensional subspace has the familiar characteristic polynomial:

$$X^2 - a_p(f)X + p^{k-1}\varepsilon(p)$$

where ε is the nebentypus of f. If α and β are roots of this polynomial, with $\alpha \neq \beta$, then the p-stabilized eigenforms $f_\alpha := f(z) - \alpha f(pz)$ and $f_\beta := f(z) - \beta f(pz)$ form a basis for this subspace of oldforms, such that $U_p f_\alpha = \beta f_\alpha$ and $U_p f_\beta = \alpha f_\beta$. Let $\phi_f, \phi_{f_\alpha}, \phi_{f_\beta}$ denote the automorphic forms associated to f, f_α, f_β in $\pi = \pi_f$. Let $\phi_{f_\alpha,p}$ and $\phi_{f_\beta,p}$ denote the components of ϕ_{f_α} and ϕ_{f_β} inside π_p. They are fixed by the following subgroup:

$$K_0(p) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod{p\mathbb{Z}_p} \right\}.$$

This admits an Iwahori decomposition with respect to $B = TN$:

$$K_0(p) = \left(\begin{smallmatrix} 1 & \\ p\mathbb{Z}_p & 1 \end{smallmatrix}\right) \begin{pmatrix} \mathbb{Z}_p^\times & \\ & \mathbb{Z}_p^\times \end{pmatrix} \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{smallmatrix}\right).$$

Let $N_0 := \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{smallmatrix}\right)$, then indeed $\phi_{f_\alpha,p}, \phi_{f_\beta,p} \in \pi_p^{N_0}$. Let $K_0 := K_0(p)$. Normalize the Haar measure on $\mathrm{GL}_2(\mathbb{Z}_p)$ so that $\mathrm{meas}(K_0) = 1$, and normalize the product measure on the Iwahori decomposition of K_0 so that $\mathrm{meas}(N_0) = 1$. Recall that S^- consists of elements $s = \left(\begin{smallmatrix} s_1 & \\ & s_2 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ such that $|s_1| \leq |s_2|$. Here are some facts:

(a) The proof of [Cas95, Lemma 1.5.1] tells us that for any $s \in S^-$:

$$[K_0 : K_0 \cap sK_0s^{-1}] = [N_0 : sN_0s^{-1}]. \qquad \text{(Recall } sN_0s^{-1} \subset N_0\text{)}$$

(b) The proof of [Eme06a, Lemma 4.4.2] tells us that for any $s \in S^-$, and any $\phi \in \pi_p^{N_0}$:

$$[N_0 s N_0]\phi = [N_0 : sN_0s^{-1}]\mathcal{P}_{N_0,s}(\phi). \qquad \text{(Recall } sN_0s^{-1} \subset N_0\text{)}$$

(c) Our earlier calculation in particular implies that for any $s \in S^-$ and $\phi \in \pi_p^{K_0}$:

$$[K_0 s K_0]\phi = [K_0 : K_0 \cap sK_0s^{-1}]\mathcal{P}_{K_0,s}(\phi).$$

Therefore, $[N_0 s N_0]\phi = [K_0 s K_0]\phi$ for any $s \in S^-$ and $\phi \in \pi_p^{K_0}$ (with our choice of measures). Let $s_p = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ and $r_p := \left(\begin{smallmatrix} p & \\ & p \end{smallmatrix}\right)$ be elements of S^-. Let $U_p := [N_0 s_p N_0]$. Then by the formulas above, and the fact that we have chosen the arithmetic normalization:

$$\begin{aligned}
[N_0 s_p N_0]\phi_{f_\alpha,p} &= [K_0 s_p K_0]\phi_{f_\alpha,p} = \beta\phi_{f_\alpha,p} \\
[N_0 s_p N_0]\phi_{f_\beta,p} &= [K_0 s_p K_0]\phi_{f_\beta,p} = \alpha\phi_{f_\beta,p}
\end{aligned}$$

Since $\alpha, \beta \neq 0$, one has $\phi_{f_\alpha,p}, \phi_{f_\beta,p} \in (\pi_p)_{\mathrm{fs}}^{N_0}$ by definition. Recall that π has an adelic central character $\mathbb{A}^\times \to \mathbb{C}^\times$ given by $z \mapsto |z|^{2-k}\omega(z)$ where $\omega := \omega_\varepsilon$ is the adelic character associated to the nebentypus ε of f. Using this, one calculates:

$$\begin{aligned}
[N_0 r_p N_0]\phi_{f_\alpha,p}(g) &= [K_0 r_p K_0]\phi_{f_\alpha,p}(g) = \phi_{f_\alpha,p}(g s_p) = p^{k-2}\varepsilon(p)\phi_{f_\alpha,p}(g) \\
[N_0 r_p N_0]\phi_{f_\beta,p}(g) &= [K_0 r_p K_0]\phi_{f_\beta,p}(g) = \phi_{f_\beta,p}(g s_p) = p^{k-2}\varepsilon(p)\phi_{f_\beta,p}(g)
\end{aligned}$$

Let $\theta' := \theta_2 \otimes \theta_1$. The N-coinvariants of π_p decomposes as a T-representation into:

$$(\pi_p)_N = \delta_B^{1/2}(\theta \oplus \theta').$$

Recall that the isomorphism $(\pi_p)_{\mathrm{fs}}^{N_0} \xrightarrow{\sim} (\pi_p)_N$ commutes with the action of S^-. Since the action of S^- is invertible on $(\pi_p)_{\mathrm{fs}}^{N_0}$, this extends to a $T = \langle S^- \rangle$-equivariant isomorphism. Since the matrices s_p and r_p act on both $\phi_{f_\alpha,p}$ and $\phi_{f_\beta,p}$ via scalars, the monoid S^- must then act on them via scalars, since $\mathrm{diag}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ acts trivially on vectors in $\pi_p^{K_0}$. The characters by which S^- act on $\phi_{f_\alpha,p}$ and $\phi_{f_\beta,p}$ must be different, since $\alpha \neq \beta$. Without loss of generality, let us assume that S^- acts on $\phi_{f_\beta,p}$ via $\delta_B^{1/2}\theta$, and $\phi_{f_\alpha,p}$ via $\delta_B^{1/2}\theta'$. Calculate:

$$\begin{aligned}
p^{-1/2}\theta_1(p)\phi_{f_\beta,p} = \delta_B^{1/2}(s_p)\theta(s_p)\phi_{f_\beta,p} &= \mathcal{P}_{N_0,s_p}(\phi_{f_\beta,p}) \\
&= [N_0 : s_p N_0 s_p^{-1}]^{-1}[N_0 s_p N_0]\phi_{f_\beta,p} \\
&= p^{-1}[K_0 s_p K_0]\phi_{f_\beta,p} \\
&= p^{-1}\alpha\phi_{f_\beta,p}.
\end{aligned}$$

This implies that $\theta_1(p) = p^{-1/2}\alpha$.

$$\begin{aligned}
\theta_1(p)\theta_2(p)\phi_{f_\beta,p} = \delta_B^{1/2}(r_p)\theta(r_p)\phi_{f_\beta,p} &= \mathcal{P}_{N_0,r_p}(\phi_{f_\beta,p}) \\
&= [N_0 : r_p N_0 r_p^{-1}]^{-1}[N_0 r_p N_0]\phi_{f_\beta,p} \\
&= [K_0 r_p K_0]\phi_{f_\beta,p} \\
&= p^{k-2}\varepsilon(p)\phi_{f_\beta,p}.
\end{aligned}$$

Therefore, $\theta_1(p)\theta_2(p) = p^{k-2}\varepsilon(p)$. But $p^{k-1}\varepsilon(p) = \alpha\beta$, so $\theta_2(p) = p^{-1/2}\beta$. We can repeat the same calculation for $\phi_{f_\alpha,p}$, but it will just recover the same values for $\theta_1(p)$ and $\theta_2(p)$.

Write $\pi = \pi_\infty \otimes \pi^\infty$, $\pi^\infty = \pi_p \otimes \pi^p$, and fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. We can regard π^∞ as a representation of $\mathrm{GL}_2(\mathbb{A}^\infty)$ on a $\overline{\mathbb{Q}}_p$-vector space, and W_k as a $\overline{\mathbb{Q}}_p$-vector space equipped with an action of $\mathrm{GL}_2 / \overline{\mathbb{Q}}_p$, which restricts to an action of G. We can choose a finite extension E of $\mathbb{Q}_p$ to which π^∞ descends, and ensure the choice of E is sufficiently large so that $\mathbb{G}$ splits over E (note that this last condition is superfluous for $\mathbb{G} = \mathrm{GL}_2$). Then W_k also admits a descent to E. Let π^∞ now denote a descent of π^∞ to E, and let W_k denote a descent of W_k to E. Let us denote $\widetilde{\pi}_p := \pi_p \otimes_E W_k^\vee$ and $\widetilde{\pi}^p := \pi^p$. Then:

$$\widetilde{\pi} := \widetilde{\pi}_p \otimes_E \widetilde{\pi}^p$$

is the p-adic automorphic representation of $\mathrm{GL}_2(\mathbb{A}^\infty) = G \times \Gamma$ attached to π, obtained in the "∞-to-p switch" for cohomological automorphic representations of $\mathrm{GL}_2 / \mathbb{Q}$, which is described in more detail in [Eme06b, §3.1]; recall that the "∞-to-p switch" we described earlier only applies to groups $\mathbb{G}/\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R})$ is compact.

The p-th component of $\widetilde{\pi}_p$ of $\widetilde{\pi}$ is a locally $W_k^\vee$-algebraic representation of G. Recall that Proposition 4.47 gives us an easy formula to compute its Jacquet module. Indeed, this is:

$$J_B(\widetilde{\pi}_p) = J_B(\pi_p \otimes_E W_k^\vee) = (\pi_p)_N \otimes_E (W_k^\vee)^N = \delta_B^{1/2}(\theta \oplus \theta') \otimes_E \psi_k^\vee.$$

Let $\chi := \delta_B^{1/2}\theta\psi_k^\vee$ so that $\delta_B^{1/2}\theta$ is the unramified character associated to $\phi_{f_\beta,p}$ from earlier. Let us consider the χ-eigenspace of $J_B(\widetilde{\pi}_p)$.

Proposition 4.53. *χ is non-critical if and only if* $\mathrm{ord}_p(\alpha) < k-1$.
[Eme06b, Corollary 4.4.3]

Proof. Let $\mathbb{T}$ be the diagonal torus in $\mathbb{G} = \mathrm{GL}_2 / \mathbb{Q}$. Then $\mathbb{T} = \mathbb{S} = \mathbb{Z}_{\mathbb{M}} = \mathbb{M}$ in the definition of non-critical slope, so $Y^\bullet = X^\bullet = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. Write $X^\bullet = \mathbb{Z}^2$ so that it is spanned by characters $e_1 = [1,0] : \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto t_1$ and $e_2 = [0,1] : \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto t_2$. Let $(R^\bullet \otimes_{\mathbb{Z}} \mathbb{Q})^{\geq 0}$ be the $\mathbb{Q}_{\geq 0}$-cone in $\Delta(\mathbb{G}, \mathbb{T})$ spanned by our choice of simple root $e_1 - e_2 \in \Delta(\mathbb{G}, \mathbb{T})_s$. The corresponding simple reflection $s := s_{e_1 - e_2}$ just exchanges e_1 and e_2. Let $\rho := \frac{1}{2}(e_1 - e_2)$ denote the standard one-half sum of positive roots.

Let $h := \mathrm{ord}_p(\alpha) = \mathrm{ord}_p(p(\delta_B^{1/2}\theta)(s_p))$. From earlier, the constant term of the characteristic polynomial of U_p tells us that $k - 1 = \mathrm{ord}_p(\alpha) + \mathrm{ord}_p(\beta)$. Let $s_p' := \begin{pmatrix} 1 & \\ & p \end{pmatrix}$, then:

$$k - 1 - h = \mathrm{ord}_p(\beta) = \mathrm{ord}_p((\delta_B^{1/2}\theta)(s_p')).$$

On the other hand, since $\delta_B^{1/2}\theta$ is unramified, we have already calculated that:

$$\mathrm{slope}(\delta_B^{1/2}\theta) = [\mathrm{ord}_p((\delta_B^{1/2}\theta)(s_p)), \mathrm{ord}_p((\delta_B^{1/2}\theta)(s_p'))] = [h-1, k-1-h].$$

Finally, one calculates:

$$\begin{aligned} \mathrm{slope}(\delta_B^{1/2}\theta) + \rho + s(\psi_k^\vee + \rho) &= [h-1, k-1-h] + [\tfrac{1}{2}, -\tfrac{1}{2}] + s([0, 2-k] + [\tfrac{1}{2}, -\tfrac{1}{2}]) \\ &= [h-1, k-1-h] + [\tfrac{1}{2}, -\tfrac{1}{2}] + [2-k, 0] + [-\tfrac{1}{2}, \tfrac{1}{2}] \\ &= [h+1-k, k-1-h] \\ &= (h+1-k)[1, -1]. \end{aligned}$$

The character χ is non-critical if and only if $h + 1 - k < 0$. This is equivalent to:

$$\mathrm{ord}_p(\alpha) < k - 1. \qquad \square$$

As promised, we present the final theorem of this section.

Theorem 4.54. *Let V be an object of $\mathrm{Rep}_{\mathrm{la.c}}(G)$, and suppose that V admits a G-invariant norm. If χ is of non-critical slope, then the following map is an isomorphism:*

$$J_P(V_{W\text{-lalg}})^{\chi} \xrightarrow{\sim} J_P(V)^{\chi}_{W^{\mathbb{N}}\text{-lalg}} = (V^{N_0, Z_M^+ = \chi})_{W^{\mathbb{N}}\text{-lalg}}.$$

[Eme06a, Theorem 4.4.5], [Eme06b, Proposition 2.3.6]

5. Eigenvarieties

Let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}$. Let us also assume that $\mathbb{G}(\mathbb{R})$ is compact and connected. Suppose $\mathbb{G}$ is quasi-split over $\mathbb{Q}_p$. Let E be a finite extension of $\mathbb{Q}_p$ over which $\mathbb{G}$ splits. We fix a Levi factor $\mathbb{T}$ in a Borel subgroup $\mathbb{B}$ of $\mathbb{G}/\mathbb{Q}_p$, so that $\mathbb{T}$ is a torus, and let $\widehat{T}$ denote the rigid analytic space over E which parameterizes the locally analytic characters of $T := \mathbb{T}(\mathbb{Q}_p)$ over E. Let $B := \mathbb{B}(\mathbb{Q}_p)$, $G := \mathbb{G}(\mathbb{Q}_p)$, and $\Gamma := \pi_0 \times \mathbb{G}(\mathbb{A}^{\infty,p})$.

Let us fix a tame level K^p in $\mathbb{G}(\mathbb{A}^{\infty,p})$.

Definition 5.1. We say that K^p is *unramified* at a place $q \neq p$ of $\mathbb{Q}$ if:

(a) $\mathbb{G}$ is unramified at q; that is, if $\mathbb{G}$ is quasi-split over $\mathbb{Q}_q$, and splits over an unramified extension of $\mathbb{Q}_q$.
(b) The compact open subgroup $K^p_q := K^q \cap \mathbb{G}(\mathbb{Q}_q)$ of $\mathbb{G}(\mathbb{Q}_q)$ is a hyperspecial maximal compact subgroup of $\mathbb{G}(\mathbb{Q}_q)$.

Otherwise, we say that K^p is *ramified* at q.
[Eme06b, Definition 2.3.1]

Let S denote the (finite) set of ramified primes of K^p. If H is a compact open subgroup of a locally compact group G, then we let $\mathcal{H}(G//H)$ denote the Hecke algebra of H double cosets of G, with coefficients in E. We abbreviate $\mathcal{H}(\mathbb{G}(\mathbb{A}^{\infty,p})//K^p)$, $\mathcal{H}(\mathbb{G}(\mathbb{Q}_S)//K^p_S)$, and $\mathcal{H}(\mathbb{G}(\mathbb{A}^{\infty,S})//K^{p,S})$ by $\mathcal{H}(K^p)$, $\mathcal{H}(K^p)^{\mathrm{ram}}$, and $\mathcal{H}(K^p)^{\mathrm{sph}}$, respectively. There is a tensor product decomposition of the Hecke algebra induced by the product decomposition of K^p:

$$\mathcal{H}(K^p) = \mathcal{H}(K^p)^{\mathrm{ram}} \otimes_E \mathcal{H}(K^p)^{\mathrm{sph}}.$$

Indeed, $\mathcal{H}(K^p)^{\mathrm{sph}}$ is a central subalgebra of $\mathcal{H}(K^p)$. By Theorem 3.50, the space $\widetilde{H}^0(K^p)_{\mathrm{la}}$ is naturally identified with the K^p-invariants in the admissible locally analytic representation $\widetilde{H}^0_{\mathrm{la}}$ of $\pi_0 \times \mathbb{G}(\mathbb{A}^\infty)$. The space $\widetilde{H}^0(K^p)_{\mathrm{la}}$ is equipped with a locally analytic representation of G, together with commuting continuous actions of π_0 and $\mathcal{H}(K^p)^{\mathrm{sph}}$. Then the Jacquet module $J_B(\widetilde{H}^0(K^p)_{\mathrm{la}})$ is an essentially admissible locally analytic representation of T, again equipped with commuting continuous actions of π_0 and $\mathcal{H}(K^p)^{\mathrm{sph}}$.

Recall from the definition of essential admissibility that this means $J_B(\widetilde{H}^0(K^p)_{\mathrm{la}})'_b$ is a coadmissible module for the nuclear Fréchet algebra $\mathcal{C}^{\mathrm{an}}(\widehat{T}, E)\widehat{\otimes}_E \mathcal{D}^{\mathrm{la}}(H, E)_b$ for some open compact subgroup H of T. We choose $H = T^0$ to be the maximal compact subgroup of T. Now if E is discretely valued (as it is here) then this completed tensor product is actually a Fréchet-Stein algebra. See the paragraph just before [Eme17, Definition 6.4.9] for a discussion of this. There is the following proposition:

Proposition 5.2. *If Z is a topologically finitely generated abelian locally $\mathbb{Q}_p$-analytic group, then there is a natural continuous injection of topological E-algebras $\mathcal{D}^{\mathrm{la}}(Z, E)_b \to \mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$. This map has dense image, and if Z is furthermore compact, then it is an isomorphism.*
[Eme17, Proposition 6.4.6]

Since T^0 is compact, $\mathcal{D}^{\mathrm{la}}(T^0, E)_b \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\widehat{T^0}, E)$, and hence the action of:

$$\mathcal{C}^{\mathrm{an}}(\widehat{T}, E)\widehat{\otimes}_E \mathcal{D}^{\mathrm{la}}(T^0, E)_b \cong \mathcal{C}^{\mathrm{an}}(\widehat{T}, E)\widehat{\otimes}_E \mathcal{C}^{\mathrm{an}}(\widehat{T^0}, E)$$

factors through the quotient:

$$\mathcal{C}^{\mathrm{an}}(\widehat{T}, E)\widehat{\otimes}_{\mathcal{C}^{\mathrm{an}}(\widehat{T^0}, E)} \mathcal{C}^{\mathrm{an}}(\widehat{T^0}, E) \cong \mathcal{C}^{\mathrm{an}}(\widehat{T}, E) = \mathcal{O}_{\widehat{T}}(\widehat{T}).$$

Recall that $\widehat{T}$ is a rigid analytic variety defined over E. Indeed, the construction of $\widehat{T}$ in [Eme17, Proposition 6.4.5] tells us that $\widehat{T}$ is a quasi-Stein rigid space. Then the theory of quasi-Stein rigid spaces tells us the following:

Theorem 5.3. *There is an equivalence of categories between the category of coherent rigid analytic sheaves on the quasi-Stein space $\widehat{T}$ and the category of coadmissible modules over the Fréchet-Stein algebra $\mathcal{O}_{\widehat{T}}(\widehat{T}) = \mathcal{C}^{\mathrm{an}}(\widehat{T}, E)$. This equivalence is given by sending a coherent sheaf to its module of global sections.*
[ST04, Corollary 23.6]

We can extend this equivalence even further.

Proposition 5.4. *There is an anti-equivalence of categories between the category of coherent sheaves on $\widehat{T}$, and the category of essentially admissible locally analytic representation of T defined over E. This equivalence is given by sending a coherent sheaf over $\widehat{T}$ to the strong dual of its module of global sections over $\mathcal{C}^{\mathrm{an}}(\widehat{T}, E)$, which can be viewed as a representation of T via the natural embedding $T \to \mathcal{C}^{\mathrm{an}}(\widehat{T}, E)$.*
[Eme06a, Proposition 2.3.2]

Let $\mathcal{M}$ denote the coherent sheaf on $\widehat{T}$ associated to the essentially admissible locally analytic representation $J_B(\widetilde{H}^0(K^p)_{\mathrm{la}})$ of T. Since $\mathcal{M}$ is an $\mathcal{O}_{\widehat{T}}$-module, we can treat $\underline{\mathrm{End}}(\mathcal{M})$ as an $\mathcal{O}_{\widehat{T}}$-algebra, with the natural structure map $\mathcal{O}_{\widehat{T}} \to \underline{\mathrm{End}}(\mathcal{M})$. Recall there is also an action of the commutative E-algebra $\mathcal{H}(K^p)^{\mathrm{sph}}$ on $J_B(\widetilde{H}^0(K^p)_{\mathrm{la}})$ which induces an action of $\mathcal{H}(K^p)^{\mathrm{sph}}$ on $\mathcal{M}$ (that is, an action of $\mathcal{H}(K^p)^{\mathrm{sph}}$ on $\mathcal{M}(U)$ for all open sets $U \subset \widehat{T}$). This gives rise to a coherent $\mathcal{O}_{\widehat{T}}$-subalgebra $\mathcal{A}$ of $\underline{\mathrm{End}}(\mathcal{M})$, generated by the image of $\mathcal{H}(K^p)^{\mathrm{sph}}$. We can then form the relative rigid analytic spectrum of $\mathcal{A}$ over $\widehat{T}$, with structure map:

$$\mathcal{E} := \underline{\mathrm{Sp}}(\mathcal{A}) \to \widehat{T}.$$

This $\mathcal{E}$ is the *eigenvariety* associated to $\mathbb{G}$. It should parameterize eigenvectors for the simultaneous actions of T and $\mathcal{H}(K^p)^{\mathrm{sph}}$ on $J_B(\widetilde{H}^0(K^p)_{\mathrm{la}})$. This is precisely the content of the following key theorem.

Theorem 5.5.

(a) The natural projection $\mathcal{E} \to \widehat{T}$ is a finite morphism, with set-theoretic image equal to the support $\mathrm{Supp}(\mathcal{M})$ *of $\mathcal{M}$.*
(b) The map $\mathcal{E} \to \mathfrak{t}^\vee$, where $\mathfrak{t}^\vee$ is the dual of the Lie algebra of T, induced by composition of the structure map $\mathcal{E} \to \widehat{T}$ with the natural map $\widehat{T} \to \mathfrak{t}^\vee$, given by differentiating a character $T \to E^\times$ to obtain $\mathfrak{t} \to E$, has discrete fibres. In particular, the dimension of $\mathcal{E}$ is at most equal to the dimension of $\widehat{T}$.

(c) *By construction, there is a Zariski-closed embedding $\mathcal{E} \hookrightarrow \widehat{T} \times \mathrm{Sp}(\mathcal{H}(K^p)^{\mathrm{sph}})$. We can view $\mathcal{M}$ as a sheaf on $\mathcal{E}$ by pulling back along the structure map $\mathcal{E} \to \widehat{T}$, and then view it as a sheaf on $\widehat{T} \times \mathrm{Sp}(\mathcal{H}(K^p)^{\mathrm{sph}})$ via extension-by-zero along the Zariski-closed embedding.*

$$\widehat{T} \xleftarrow{f} \mathcal{E} \xhookrightarrow{g} \widehat{T} \times \mathrm{Sp}(\mathcal{H}(K^p)^{\mathrm{sph}})$$

Then the fibre of $g_! f^* \mathcal{M}$ over a point (χ, λ) of $\widehat{T} \times \mathrm{Sp}(\mathcal{H}(K^p)^{\mathrm{sph}})$ is the dual to the $(T = \chi, \mathcal{H}(K^p)^{\mathrm{sph}} = \lambda)$-eigenspace of $J_B(\widetilde{H}^0(K^p)_{\mathrm{la}})$. In particular, the point (χ, λ) lies in $\mathcal{E}$ if and only if this eigenspace is non-zero.

[Eme06a, Proposition 2.3.3]

Remark 5.6. As pointed out to me, $\mathcal{H}(K^p)^{\mathrm{sph}}$ is not a finitely generated E-algebra, and hence it does not make sense to write $\mathrm{Sp}(\mathcal{H}(K^p)^{\mathrm{sph}})$. However, I suspect this issue has a simple solution which is to replace "$\mathrm{Sp}(\mathcal{H}(K^p)^{\mathrm{sph}})$" with $\mathrm{Sp}(A)$ where A is the image of $\mathcal{H}(K^p)^{\mathrm{sph}}$ inside the endomorphism ring of $J_B(\widetilde{H}^0(K^p)_{\mathrm{la}})'_b$.

Example 5.7. Let M be a finite-dimensional $\mathbb{C}$-vector space, equipped with two commuting linear operators Y and X, where X is invertible, and both X and Y are diagonalizable. In particular, X and Y are simultaneously diagonalizable. For simplicity, we also assume that all eigenvalues are distinct. Thus, if m_X and m_Y are the minimal polynomials of X and Y, respectively, and if $e_1, \dots, e_n$ is a basis of simultaneous eigenvectors for X and Y, then:

$$\begin{aligned} Xe_i &= \mu_i e_i \qquad \text{for all } 1 \le i \le n, \\ Ye_i &= \lambda_i e_i \qquad \text{for all } 1 \le i \le n, \\ m_X(t) &= (t - \mu_1) \dots (t - \mu_n), \\ m_Y(t) &= (t - \lambda_1) \dots (t - \lambda_n). \end{aligned}$$

Fix an isomorphism $\mathrm{End}_{\mathbb{C}}(M) \cong M_n(\mathbb{C})$ induced by this choice of basis. Let us rephrase everything in terms of our eigenvariety setup:

(i) We can view M as a module for $\mathbb{C}[X, X^{-1}] = \mathcal{O}_{\widehat{T}}(\widehat{T})$, where $T = \mathbb{Z}$, $\widehat{T} = \mathbb{G}_m$, and the abelian group T acts on M via the inclusion $\mathbb{Z} \to \mathbb{C}[X, X^{-1}]$ sending $1 \mapsto X$. In particular, this gives us a map $\mathbb{C}[X, X^{-1}] \to \mathrm{End}_{\mathbb{C}}(M)$, with image:
$$\frac{\mathbb{C}[X, X^{-1}]}{(m_X(X))}.$$

(ii) We can also endow M with an action of $\mathcal{H} = \mathbb{C}[Y]$, which is the analogue of the Hecke algebra in our setup. This induces a natural map $\mathbb{C}[Y] \to \mathrm{End}_{\mathbb{C}}(M)$, with image:
$$\frac{\mathbb{C}[Y]}{(m_Y(Y))}.$$

The image of $\mathbb{C}[Y]$ commutes with the image of $\mathbb{C}[X, X^{-1}]$ in $\mathrm{End}_{\mathbb{C}}(M)$. Let A denote the $\mathbb{C}[X, X^{-1}]$-algebra in $\mathrm{End}_{\mathbb{C}}(M)$ generated by the image of $\mathbb{C}[Y]$. Then:

$$A = \frac{\mathbb{C}[X, X^{-1}, Y]}{\sum_S (m_S(X, Y))}$$

where for a subset $S \subset \{1, \dots, n\}$, one defines:

$$m_S(X, Y) := \prod_{i \in S} (X - \mu_i) \prod_{i \notin S} (Y - \lambda_i).$$

There is a natural chain of ring maps:

$$\mathbb{C}[X,X^{-1},Y] \twoheadrightarrow \frac{\mathbb{C}[X,X^{-1},Y]}{\sum_S(m_S(X,Y))} \leftarrow \frac{\mathbb{C}[X,X^{-1}]}{(m_X(X))} \twoheadleftarrow \mathbb{C}[X,X^{-1}].$$

This induces a map on spectra:

$$\widehat{T} \times \mathrm{Spec}(\mathcal{H}) \hookleftarrow \mathcal{E} := \mathrm{Spec}(A) \to \mathrm{Supp}(M) \hookrightarrow \widehat{T}.$$

The maximal ideals of A are in one-to-one correspondence with those maximal ideals of $\mathbb{C}[X,X^{-1},Y]$ that contain $\sum_S(m_S(X,Y))$. One can check that these ideals are exactly $\mathfrak{m}_i = (X-\mu_i)+(Y-\lambda_i)$ for $1 \leq i \leq n$. So the preimage of the maximal ideal $(X-\mu_i)$ under the structure map $\mathcal{E} \to \widehat{T}$ contains a unique maximal ideal $\mathfrak{m}_i = (X-\mu_i)+(Y-\lambda_i)$.

Furthermore, we can endow M with an A-module structure via base change: $M \otimes_{\mathbb{C}[X,X^{-1}]} A$. This extends to a natural action of $\mathbb{C}[X,X^{-1},Y]$ by pulling back along the surjection. Let $(X-\mu_i, Y-\lambda_i)$ denote some maximal ideal of A, then the fibre of this $\mathbb{C}[X,X^{-1},Y]$-module at the maximal ideal that we have selected can be written as:

$$M \otimes_{\mathbb{C}[X,X^{-1}]} \frac{\mathbb{C}[X,X^{-1},Y]}{(X-\mu_i, Y-\lambda_i)}.$$

This is a quotient of M, isomorphic to the $(X=\mu_i, Y=\lambda_i)$-subspace of M.

Remark 5.8. If M is an R-module, and $\mathcal{H}$ is a commutative ring of R-linear endomorphisms of M, then let $\mathcal{T}$ denote the image of $\mathcal{H}$ in $\mathrm{End}_R(M)$, which [Bel21] calls an *eigenalgebra*. The properties of $\mathrm{Spec}(\mathcal{T})$ are studied extensively in [Bel21, Chapter I].

Example 5.9. Let $\Pi := \widetilde{H}^0(K^p)_{\mathrm{la}}$. Let $\mathcal{E} \to \widehat{T}$ be the structure map of the eigenvariety $\mathcal{E}$ associated to the essentially admissible locally analytic representation $J_B(\Pi)$ of T. We give an explicit description of the fibres of the structure map $\mathcal{E} \to \widehat{T}$. For a point $x \in \widehat{T}$, let $k(x)$ denote its residue field, and associate to x the following character, obtained by composing the natural embedding $T \to \mathcal{O}_{\widehat{T}}(\widehat{T})$ with the reduction map at x:

$$\chi_x : T \to \mathcal{O}_{\widehat{T}}(\widehat{T}) \to \mathcal{O}_{\widehat{T}}(\widehat{T}) \otimes k(x).$$

Let $\mathcal{M}$ be the sheaf on $\widehat{T}$ associated to the representation $J_B(\Pi)$ of T. Then

$$\begin{aligned}
\mathcal{M}_x \otimes k(x) &= \mathcal{M}(\widehat{T}) \otimes k(x) && \text{(since } \widehat{T} \text{ is quasi-Stein)} \\
&= J_B(\Pi)' \otimes k(x) \\
&= \mathcal{L}_{b,T^+}(\mathcal{O}_{\widehat{T}}(\widehat{T}), \Pi^{N_0})' \otimes k(x) \\
&= [\mathcal{O}_{\widehat{T}}(\widehat{T}) \otimes_{E[T^+]} (\Pi^{N_0})'] \otimes k(x) \\
&= [k(x) \otimes \mathcal{O}_{\widehat{T}}(\widehat{T})] \otimes_{E[T^+]} (\Pi^{N_0})' \\
&= \chi_x \otimes_{E[T^+]} (\Pi^{N_0})' \\
&= \mathrm{Hom}_{T^+}(\chi_x, \Pi^{N_0})'.
\end{aligned}$$

On the other hand, let $U := \mathrm{Sp}(B) \subset \widehat{T}$ be an affinoid containing x. Then the fibre at x is:

$$\begin{aligned}\underline{\mathrm{Sp}}(\mathcal{A})_x &:= \underline{\mathrm{Sp}}(\mathcal{A}) \times_{\widehat{T}} \mathrm{Sp}(k(x)) \\ &= \mathrm{Sp}(\mathcal{A}(U)) \times_U \mathrm{Sp}(k(x)) \\ &= \mathrm{Sp}(\mathcal{A}(U) \otimes_B k(x)) \\ &= \mathrm{Sp}(\mathcal{A}_x \otimes_{B_x} k(x)).\end{aligned}$$

Since $\mathcal{A} \subset \underline{\mathrm{End}}(\mathcal{M})$, there is an inclusion of fibres:

$$\mathcal{A}_x \otimes_{B_x} k(x) \subset \mathrm{End}(\mathcal{M}_x) \otimes_{B_x} k(x) = \mathrm{End}(\mathcal{M}_x \otimes_{B_x} k(x)).$$

It is not difficult to see that $\mathcal{A}_x \otimes_{B_x} k(x)$ is just the subring of endomorphisms of $\mathcal{M}_x \otimes_{B_x} k(x)$, which we interpret to be the dual of the χ_x-isotypic component of Π^{N_0}, generated by the induced action of $\mathcal{H}(K^p)^{\mathrm{sph}}$ on this space. Therefore, $\underline{\mathrm{Sp}}(\mathcal{A})_x$ encodes the eigenvectors in Π^{N_0} simultaneous for the action of $\mathcal{H}(K^p)^{\mathrm{sph}}$ via any character, and of T via χ_x.

In order to work with the eigenvariety in practice, one might wish to define some auxiliary spaces, such as the weight space and spectral variety, which we shall define promptly.

Let T^0 denote the maximal compact subgroup of T. We define

$$\mathcal{W} := \widehat{T^0}$$

to be the *weight space* of our eigenvariety. Choose a distinguished element $z \in T^+$ such that T is generated as a monoid by T^+ and the group generated by z. Let Y be the closed subgroup of T generated by T^0 and z. Let $\widehat{Y}$ denote the character variety of Y. Then $\widehat{Y}$ can be identified with $\mathcal{W} \times \mathbb{G}_m$:

$$\widehat{Y} = \widehat{T^0} \times \widehat{\langle z \rangle} \cong \mathcal{W} \times \mathbb{G}_m.$$

If we denote the global sections of $\mathbb{G}_m$ by $\mathbb{C}\{\!\{X, X^{-1}\}\!\}$, then we remark crucially that the isomorphism on the right hand side is induced by the map on global sections which sends $\delta_z \mapsto X^{-1}$ (and not X), where δ_z is the evaluation-at-z function.

Let Y^+ be the submonoid of $Y \cap T^+$ generated by T^0 and z. Then Y^+ is open in Y, and Y^+ generates Y as a group. Furthermore, $YT^+ = T$.

Recall that $\widetilde{H}^0(K^p)_{\mathrm{la}}$ is a representation of G, and $\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0}$ admits an action of T^+. The functor of taking finite-slope depends on the action of T^+, since intuitively we are finding a maximal subspace on which the action of T^+ becomes invertible. Let us equip $\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0}$ with an action of Y^+ via restriction. Then there is a natural Y-equivariant inclusion map:

$$J_B(\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0}) := (\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0})_{T\text{-fs}} \to (\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0})_{Y\text{-fs}}.$$

By [Eme06a, Lemma 3.2.22], one has that the T^+-action on $\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0}$ induces a T^+-action on its finite slope part $(\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0})_{Y\text{-fs}}$, which then comes with a natural action of $YT^+ = T$. The Y-equivariant map then upgrades to a T-equivariant map for free. Finally, the result of [Eme06a, Proposition 3.2.27] tells us that the above map is a T-equivariant isomorphism:

$$J_B(\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0}) := (\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0})_{T\text{-fs}} \xrightarrow{\sim} (\widetilde{H}^0(K^p)_{\mathrm{la}}^{N_0})_{Y\text{-fs}}.$$

Recall that $J_B : \mathrm{Rep}_{\mathrm{es}}(G) \to \mathrm{Rep}_{\mathrm{es}}(T)$, so the left hand side is an essentially admissible locally analytic representation of T. A crucial part of showing that the target of J_B is $\mathrm{Rep}_{\mathrm{es}}(T)$ relies on the following pair of lemmas. (We ultimately want to show that the right hand side is an essentially admissible locally analytic representation of Y.) We digress briefly to discuss some generalities, before returning to our current discussion.

Let A be an E-Fréchet algebra, written as a projective limit $A = \varprojlim_n A_n$, where each A_n is a compact type topological E-algebra, each of the transition maps $A_{n+1} \to A_n$ is compact, and each of the natural projection maps $A \to A_n$ has dense image. Suppose also that the projective system is cofinal with a projective system of E-Banach algebras, so that A is a nuclear Fréchet algebra, in the sense of [Eme17, Definition 1.2.12].

Lemma 5.10. *Let V be a locally convex E-vector space of compact type, equipped with an A-module structure for which the multiplication map $A \times V \to V$ is separately continuous. Suppose also that V is equipped with a topological Z^+-action commuting with the above given A-action, so that V is an object of* $\mathrm{Rep}_{\mathrm{top.c}}(Z^+)$*. Then the A-module structure on V_{fs} induced by functoriality makes V_{fs} into an A-module, and the multiplication map $A \times V_{\mathrm{fs}} \to V_{\mathrm{fs}}$ is again separately continuous.*
[Eme06a, Lemma 3.2.22]

In the context of the preceding lemma, the A-module structure on V_{fs} induces a topological A-module structure on its strong dual $(V_{\mathrm{fs}})'_b$ [Eme17, Proposition 1.2.14]. Since V_{fs} is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(Z)$, the Z-action on $(V_{\mathrm{fs}})'_b$ extends uniquely to a topological $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)$-action [Eme17, Proposition 6.4.7]. Therefore, $(V_{\mathrm{fs}})'_b$ is in fact a topological $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)\widehat{\otimes}_E A$-module.

Proposition 5.11. *In the above situation, suppose given the following data:*

(i) For each $n \geq 0$, a compact type topological A_n-module U_n, equipped with an A_n-linear action of Z^+, as well as an $A_{n+1}[Z^+]$-linear transition map $U_{n+1} \to U_n$, such that the induced $A_n[Z^+]$-linear map $A_n\widehat{\otimes}_{A_{n+1}}U_{n+1} \to U_n$ is A_n-compact, in the sense of [Eme06a, Definition 2.3.3].

(ii) An element $z \in Z^+$, such that for each $n \geq 0$, the map $U_n \to U_n$ induced by z factors through the transition map $A_n\widehat{\otimes}_{A_{n+1}}U_{n+1} \to U_n$, so as to give a commutative diagram:

$$\begin{tikzcd} A_n\widehat{\otimes}_{A_{n+1}}U_{n+1} \arrow[r] \arrow[d, "\mathrm{id}\,\widehat{\otimes} z"'] & U_n \arrow[d, "z"] \arrow[dl] \\ A_n\widehat{\otimes}_{A_{n+1}}U_{n+1} \arrow[r] & U_n. \end{tikzcd}$$

(iii) An $A[Z^+]$-equivariant isomorphism $V'_b \xrightarrow{\sim} \varprojlim_n U_n$.

Then $(V_{\mathrm{fs}})'_b$ is a coadmissible $\mathcal{C}^{\mathrm{an}}(\widehat{Z}, E)\widehat{\otimes}_E A$-module, in the sense of [Eme17, Definition 1.2.8].
[Eme06a, Proposition 3.2.23]

In our situation, let A denote the E-Fréchet-Stein algebra:

$$A := \mathcal{D}^{\mathrm{la}}(T^0, E) \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(\widehat{T^0}, E) = \mathcal{O}_{\widehat{T^0}}(\widehat{T^0}) = \mathcal{O}_{\mathcal{W}}(\mathcal{W}).$$

Let Π be an object of $\mathrm{Rep}_{\mathrm{es}}(G)$. We have seen that Π^{N_0} comes equipped with a T^+-action, which we restrict to a Y^+-action, so that Π^{N_0} is an object of $\mathrm{Rep}_{\mathrm{top.c}}(Y^+)$. Taking finite slope gives us $(\Pi^{N_0})_{Y\text{-fs}}$ which is an object of $\mathrm{Rep}^z_{\mathrm{la.c}}(Y)$. Indeed, the Y-action on $((\Pi^{N_0})_{Y\text{-fs}})'_b$ extends uniquely to a $\mathcal{C}^{\mathrm{an}}(\widehat{Y}, E)$-action. Moreover, recall there is a natural $\mathcal{D}^{\mathrm{la}}(Y, E)_b$-module structure on $((\Pi^{N_0})_{Y\text{-fs}})'_b$, which we restrict to an action of $A := \mathcal{D}^{\mathrm{la}}(T^0, E)$. This equips the space $((\Pi^{N_0})_{Y\text{-fs}})'_b$ with a $\mathcal{C}^{\mathrm{an}}(\widehat{Y}, E)\widehat{\otimes}_E A$-module structure.

Proposition 5.12. *There is an admissible cover of $\mathcal{W}$ by open affinoids*

$$U_1 \subset U_2 \subset \cdots \subset U_h \subset \ldots$$

such that for each $h \geq 1$, there exist(s):

(i) An $A_h := \mathcal{O}_{\mathcal{W}}(U_h)$-Banach module V_h satisfying condition (Pr) *of* [Buz07].

(ii) An A_h-compact endomorphism, denoted z_h, of the A_h-module V_h. (Recall the notion of A_h-compact endomorphisms of spaces of (Pr)*-type from* [Buz07, §2].*)*

(iii) Continuous A_h-linear maps:

$$\alpha_h : V_h \to V_{h+1} \widehat{\otimes}_{A_{h+1}} A_h$$
$$\beta_h : V_{h+1} \widehat{\otimes}_{A_{h+1}} A_h \to V_h$$

such that $\beta_h \circ \alpha_h = z_h$ and $\alpha_h \circ \beta_h = z_{h+1} \otimes 1_{A_h}$ where β_h is A_h-compact.

(iv) An $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$-linear topological isomorphism:

$$(\Pi^{N_0})'_b \cong \varprojlim_h V_h$$

commuting with the actions of z (on the left side) and $(z_h)_{h \geq 1}$ (on the right side).

These conditions can be summarized in the following commutative diagram.

$$\begin{tikzcd}
(\Pi^{N_0})' \arrow[r] \arrow[d, "z"] & \dots \arrow[r] & V_{h+1} \arrow[r] \arrow[d, "z_{h+1}"] & V_{h+1} \widehat{\otimes}_{A_{h+1}} A_h \arrow[r, "\beta_h"] \arrow[d, "z_{h+1} \otimes 1_{A_h}"] & V_h \arrow[r] \arrow[d, "z_h"] \arrow[dl, "\alpha_h"'] & \dots \\
(\Pi^{N_0})' \arrow[r] & \dots \arrow[r] & V_{h+1} \arrow[r] & V_{h+1} \widehat{\otimes}_{A_{h+1}} A_h \arrow[r, "\beta_h"] & V_h \arrow[r] & \dots
\end{tikzcd}$$

[BHS17, Proposition 5.3]

This is precisely the construction required to imply that $((\Pi^{N_0})_{Y\text{-fs}})'_b$ is a coadmissible $\mathcal{C}^{\mathrm{an}}(\widehat{Y}, E) \widehat{\otimes}_E A$-module. This furthermore implies that $(\Pi^{N_0})_{Y\text{-fs}}$ is an essentially admissible locally analytic representation of Y.

Therefore, returning to the situation that we are interested in, we have shown that the locally analytic Y-representation $(\widetilde{H}^0(K^p)^{N_0}_{\mathrm{la}})_{Y\text{-fs}}$ is actually an essentially admissible locally analytic representation of Y. By imitating the arguments earlier in this section, we can find a coherent sheaf $\mathcal{N}$ on $\widehat{Y} = \mathcal{W} \times \mathbb{G}_m$ with global section:

$$\Gamma(\widehat{Y}, \mathcal{N}) = ((\widetilde{H}^0(K^p)^{N_0}_{\mathrm{la}})_{Y\text{-fs}})'_b \cong ((\widetilde{H}^0(K^p)^{N_0}_{\mathrm{la}})_{T\text{-fs}})'_b = J_B(\widetilde{H}^0(K^p)^{N_0}_{\mathrm{la}})'_b.$$

Let $\mathcal{Y}_z$ denote the support of $\mathcal{N}$ in $\mathcal{W} \times \mathbb{G}_m$. The following square commutes.

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\kappa"] & \widehat{T} \arrow[d] & \\
\mathcal{Y}_z \arrow[r, hook, "h"] \arrow[rr, bend right, "g"'] & \mathcal{W} \times \mathbb{G}_m \arrow[r] & \mathcal{W}.
\end{tikzcd}$$

Indeed, one can show that $\mathcal{Y}_z$ is the image of $\mathcal{E}$ under the composition $\mathcal{E} \to \widehat{T} \to \mathcal{W} \times \mathbb{G}_m$. This in particular implies that the arrow $\mathcal{E} \to \mathcal{Y}_z$ is surjective.

Definition 5.13. A closed analytic subvariety, in the sense of [BGR84, §9.5.3], of $\mathcal{W} \times \mathbb{G}_m$ is called a *Fredholm hypersurface* if it is of the form $Z(F)$ for $F \in 1 + T\mathcal{O}(\mathcal{W})\{\!\{T\}\!\}$, where

$$Z(F) := \{(x, t) \in \mathcal{W} \times \mathbb{G}_m : F(x, t^{-1}) = 0\},$$

and $\mathcal{O}(\mathcal{W})\{\{T\}\} = \mathcal{O}(\mathcal{W} \times \mathbb{A}^1)$ denotes the ring of convergent power series on $\mathcal{W} \times \mathbb{A}^1$. Please refer to [CM98, §1.3], [Buz07, §3], [Con99, §4] and [Che04, §5,§6] for more details.

Proposition 5.14. *The space $\mathcal{Y}_z$ is a Fredholm hypersurface of $\mathcal{W} \times \mathbb{G}_m$. Furthermore, there exists an admissible cover $\{U_i'\}_{i\in I}$ of $\mathcal{Y}_z$ by affinoids U_i' for which g induces a finite surjective morphism from U_i' onto an open affinoid W_i of $\mathcal{W}$, and U_i' is a connected component of $g^{-1}(W_i)$. Finally, for each $i \in I$, the section $\Gamma(U_i', \mathcal{N})$ is a finite projective $\mathcal{O}_{\mathcal{W}}(W_i)$-module.* [BHS17, Lemma 3.9]

Proof. Let $\Pi := \widetilde{H}^0(K^p)_{\mathrm{la}}$. We use the admissible cover $\{U_h\}_{h\geq 1}$ of $\mathcal{W}$ constructed earlier, such that $(\Pi^{N_0})_b' = \varprojlim_h V_h$. Recall that $z_h = \beta_h \circ \alpha_h$ and $z_{h+1} \otimes 1_{A_h} = \alpha_h \circ \beta_h$, and hence their characteristic power series give the same element of $A_h\{\{T\}\}$ by [Buz07, Lemma 2.12]. Furthermore, [Buz07, Lemma 2.13] tells us that the image of the characteristic power series of z_{h+1} agrees with that of $z_{h+1} \otimes 1_{A_h}$ under the natural map $A_{h+1}\{\{T\}\} \to A_h\{\{T\}\}$. Let F_h denote the characteristic power series of z_h, then we have just shown that the image of $F_{h+1} \in A_{h+1}\{\{T\}\}$ in $A_h\{\{T\}\}$ coincides with F_h. Let F denote the compatible system $(F_h)_{h\geq 1}$ which we can view as an element of $\mathcal{O}_{\mathcal{W}}(\mathcal{W})\{\{T\}\}$.

Recall that $\mathcal{Y}_z$ is the support of $\mathcal{N}$. We shall show that $\mathcal{Y}_z$ consists precisely of the points $(y, \lambda) \in \mathcal{W} \times \mathbb{G}_m$ such that $F(y, \lambda^{-1}) = 0$. Let us henceforth fix $x = (y, \lambda) \in \mathcal{W} \times \mathbb{G}_m$, and $h \geq 1$ such that $y \in U_h$.

Lemma 5.15. *The fibre $\mathcal{N}_x \neq 0$ if and only if $\Gamma(U_h \times \mathbb{G}_m, \mathcal{N}) \otimes_{\mathcal{O}(U_h\times\mathbb{G}_m)} k(x) \neq 0$, where $k(x) := \mathcal{O}_{\mathcal{W}\times\mathbb{G}_m,x}/\mathfrak{m}_x$ is the residue field of $\mathcal{W} \times \mathbb{G}_m$ at x.*

Proof. Since $U_h \times \mathbb{G}_m$ is a quasi-Stein space, choose an increasing chain of open affinoid neighbourhoods $\{X_j\}_{j\geq 1}$ in $U_h \times \mathbb{G}_m$ and an isomorphism:

$$U_h \times \mathbb{G}_m \xrightarrow{\sim} \bigcup_{j\geq 1} X_j$$

which realizes the quasi-Stein structure on $U_h \times \mathbb{G}_m$. Recall from the remarks following [Eme17, Definition 2.1.18] that $\mathcal{O}(U_h \times \mathbb{G}_m)$ is a Fréchet-Stein algebra. This enables us to rewrite the global sections of $\mathcal{N}$ into a projective limit:

$$\Gamma(U_h \times \mathbb{G}_m, \mathcal{N}) = \varprojlim_j \Gamma(X_j, \mathcal{N}).$$

Since each X_j is affinoid, it is a fact of commutative algebra that:

$$\Gamma(X_j, \mathcal{N}) \otimes_{\mathcal{O}(X_j)} \mathcal{O}_{\mathcal{W}\times\mathbb{G}_m,x} \xrightarrow{\sim} \mathcal{N}_x.$$

By commuting the projective limit and the tensor product, we obtain:

$$\begin{aligned}
\Gamma(U_h \times \mathbb{G}_m, \mathcal{N}) \otimes_{\mathcal{O}(U_h\times\mathbb{G}_m)} \mathcal{O}_{\mathcal{W}\times\mathbb{G}_m,x} &= \Big(\varprojlim_j \Gamma(X_j, \mathcal{N})\Big) \otimes_{\mathcal{O}(U_h\times\mathbb{G}_m)} \mathcal{O}_{\mathcal{W}\times\mathbb{G}_m,x} \\
&= \varprojlim_j \Big(\Gamma(X_j, \mathcal{N}) \otimes_{\mathcal{O}(X_j)} \mathcal{O}_{\mathcal{W}\times\mathbb{G}_m,x}\Big) \\
&= \varprojlim_j \mathcal{N}_x \\
&= \mathcal{N}_x.
\end{aligned}$$

This is an isomorphism of $\mathcal{O}_{\mathcal{W}\times\mathbb{G}_m,x}$-modules. Let $\mathfrak{m}_x$ denote the maximal ideal in $\mathcal{O}_{\mathcal{W}\times\mathbb{G}_m,x}$. Then taking the reduction modulo $\mathfrak{m}_x$ on both sides gives:

$$\Gamma(U_h\times\mathbb{G}_m,\mathcal{N})\otimes_{\mathcal{O}(U_h\times\mathbb{G}_m)}k(x)=\mathcal{N}_x/\mathfrak{m}_x\mathcal{N}_x.$$

The conclusion of this lemma follows, because $\mathcal{N}_x=0$ if and only if $\mathcal{N}_x/\mathfrak{m}_x\mathcal{N}_x=0$ as a consequence of Nakayama's lemma. $\square$

Let us compute $\Gamma(U_h\times\mathbb{G}_m,\mathcal{N})$ more explicitly.

$$\begin{aligned}
\Gamma(U_h\times\mathbb{G}_m,\mathcal{N})&=\Gamma(\mathcal{W}\times\mathbb{G}_m,\mathcal{N})\widehat{\otimes}_{\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)}\mathcal{O}(U_h\times\mathbb{G}_m) &&\text{([BHS17, Lemma 5.5])}\\
&=J_B(\Pi)'\widehat{\otimes}_{\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)}\mathcal{O}(U_h\times\mathbb{G}_m)\\
&=(\Pi^{N_0})'\widehat{\otimes}_{E[Y^+]}\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)\widehat{\otimes}_{\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)}\mathcal{O}(U_h\times\mathbb{G}_m)\\
&=(\Pi^{N_0})'\widehat{\otimes}_{E[Y^+]}\mathcal{O}(U_h\times\mathbb{G}_m)\\
&=(\Pi^{N_0})'\widehat{\otimes}_{\mathcal{O}(\mathcal{W})[z]}\mathcal{O}(U_h)\{\!\{z,z^{-1}\}\!\}\\
&=\Big(\varprojlim_{h'}V_{h'}\Big)\widehat{\otimes}_{\mathcal{O}(\mathcal{W})[z]}\mathcal{O}(U_h)\{\!\{z,z^{-1}\}\!\}\\
&=\varprojlim_{h'\geq h}\Big(V_{h'}\widehat{\otimes}_{\mathcal{O}(U_{h'})[z]}\mathcal{O}(U_h)\{\!\{z,z^{-1}\}\!\}\Big)\\
&=\varprojlim_{h'\geq h}\Big(V_{h'}\widehat{\otimes}_{A_{h'}[z]}A_h\{\!\{z,z^{-1}\}\!\}\Big).
\end{aligned}$$

Lemma 5.16. $\Gamma(U_h\times\mathbb{G}_m,\mathcal{N})\otimes_{\mathcal{O}(U_h\times\mathbb{G}_m)}k(x)\neq 0$ *if and only if* $V_h/((z_h-\lambda)V_h+\mathfrak{p}_yV_h)\neq 0$, *where* $\mathfrak{p}_y$ *is the maximal ideal associated to the point* y *in* $\mathcal{W}$.

Proof. We calculate the fibre of $\mathcal{N}$ at x explicitly:

$$\begin{aligned}
\Gamma(U_h\times\mathbb{G}_m,\mathcal{N})\otimes_{\mathcal{O}(U_h\times\mathbb{G}_m)}k(x)&=\varprojlim_{h'\geq h}\Big(V_{h'}\widehat{\otimes}_{A_{h'}[z]}A_h\{\!\{z,z^{-1}\}\!\}\Big)\otimes_{A_h\{\!\{z,z^{-1}\}\!\}}\Big(\frac{A_h\{\!\{z,z^{-1}\}\!\}}{\mathfrak{m}_x}\Big)\\
&=\varprojlim_{h'\geq h}\Big(V_{h'}\widehat{\otimes}_{A_{h'}[z]}\Big(\frac{A_h\{\!\{z,z^{-1}\}\!\}}{\mathfrak{m}_x}\Big)\Big).
\end{aligned}$$

One direction is clear, since the first term of this projective limit is:

$$V_h\widehat{\otimes}_{A_h[z]}\Big(\frac{A_h\{\!\{z,z^{-1}\}\!\}}{\mathfrak{m}_x}\Big)=V_h/\mathfrak{m}_xV_h=\frac{V_h}{\mathfrak{p}_yV_h+(z_h-\lambda)V_h}.$$

Suppose this first term is non-zero, then recall there are maps of A_h (or $A_h[z]$)-modules:

$$\begin{tikzcd} V_h \arrow[r, "\alpha_h"] \arrow[rr, bend right, "z"'] & V_{h+1}\widehat{\otimes}_{A_{h+1}}A_h \arrow[r, "\beta_h"] & V_h. \end{tikzcd}$$

This induces natural maps of $A_h\{\!\{z,z^{-1}\}\!\}$-modules:

$$V_h\widehat{\otimes}_{A_h[z]}A_h\{\!\{z,z^{-1}\}\!\}\xrightarrow{\alpha_h}V_{h+1}\widehat{\otimes}_{A_{h+1}[z]}A_h\{\!\{z,z^{-1}\}\!\}\xrightarrow{\beta_h}V_h\widehat{\otimes}_{A_h[z]}A_h\{\!\{z,z^{-1}\}\!\}.$$

But z acts invertibly on all these spaces, so in particular we can multiply by z^{-1}:

$$\begin{tikzcd} V_h\widehat{\otimes}_{A_h[z]}A_h\{\{z,z^{-1}\}\} \arrow[r, "\alpha_h"] & V_{h+1}\widehat{\otimes}_{A_{h+1}[z]}A_h\{\{z,z^{-1}\}\} \arrow[d, "\beta_h"] \\ V_h\widehat{\otimes}_{A_h[z]}A_h\{\{z,z^{-1}\}\} \arrow[u, "z^{-1}"] \arrow[r, dashed, "\mathrm{id}"] & V_h\widehat{\otimes}_{A_h[z]}A_h\{\{z,z^{-1}\}\}. \end{tikzcd}$$

Consider the same diagram after taking reduction modulo $\mathfrak{m}_x$. Then the map $\alpha_h \circ z^{-1}$ defines a section of the first transition map. We can repeat this procedure, so that if the first term of the projective limit is non-zero, then we can lift this element to a non-zero compatible system in the projective limit itself. This completes the proof of the lemma. □

Since z_h is an A_h-compact endomorphism of V_h and it is A_h-linear, it induces a compact endomorphism of $V_h/\mathfrak{p}_yV_h$. By [Sch02, Corollary 22.9], the endomorphism $1-\lambda^{-1}z_h$ is a *Fredholm* endomorphism of $V_h/\mathfrak{p}_yV_h$ of *index* zero. (This means that its kernel and cokernel are both finite dimensional, and of the same dimension, respectively. Refer to [Sch02, §22].) Therefore, $V_h/((z_h-\lambda)V_h+\mathfrak{p}_yV_h) \neq 0$ if and only if the cokernel of $1-\lambda^{-1}z_h$ acting on $V_h/\mathfrak{p}_yV_h$ is non-zero if and only if the kernel of $1-\lambda^{-1}z_h$ acting on $V_h/\mathfrak{p}_yV_h$ is non-zero. This is just asking for λ^{-1} to be a zero of the characteristic power series of z_h on $V_h/\mathfrak{p}_yV_h$ which is an element of $k(y)\{\{T\}\}$. But one can easily check that this is the image of F_h under the map $A_h\{\{T\}\} \to k(y)\{\{T\}\}$ which is just taking the reduction of coefficients modulo $\mathfrak{p}_y$.

We have just shown that $\mathcal{Y}_z$ is a Fredholm hypersurface over $\mathcal{W}$. Thus we can leverage the general theory of Fredholm hypersurfaces. By [Buz07, Theorem 4.6], there is an admissible cover of $\mathcal{Y}_z$ by open affinoids U'_i such that the map g induces a finite surjective morphism $U'_i \to W_i$ where W_i is an open affinoid in $\mathcal{W}$ and U'_i is a connected component of $g^{-1}(W_i)$.

Fix $i \in I$. Then the first five paragraphs of [Buz07, §5] tells us that U'_i is the closed analytic subvariety of $W_i \times \mathbb{G}_m$ cut out by the ideal generated by the polynomial:

$$Q(T) \in 1 + T\mathcal{O}_{\mathcal{W}}(W_i)[T]$$

and such that $F(T) = Q(T)S(T)$ where $S(T)$ is a Fredholm series in $1+T\mathcal{O}_{\mathcal{W}}(W_i)\{\{T\}\}$ and $(Q,S)=1$. By abuse of notation, let us denote the restriction of $\mathcal{N}$ to its support $\mathcal{Y}_z$, which is most unambiguously denoted $h^*\mathcal{N}$, by $\mathcal{N}$ as well. We calculate:

$$\begin{aligned}
\Gamma(U'_i,\mathcal{N}) &:= \Gamma(U'_i, h^*\mathcal{N}) \\
&= \Gamma(\mathcal{Y}_z, h^*\mathcal{N})\widehat{\otimes}_{\mathcal{O}(\mathcal{Y}_z)}\mathcal{O}(U'_i) && \text{([BHS17, Lemma 5.5])} \\
&= \Gamma(\mathcal{W}\times\mathbb{G}_m,\mathcal{N})\widehat{\otimes}_{\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)}\mathcal{O}(\mathcal{Y}_z)\widehat{\otimes}_{\mathcal{O}(\mathcal{Y}_z)}\mathcal{O}(U'_i) \\
&= J_B(\Pi)'_b\widehat{\otimes}_{\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)}\mathcal{O}(\mathcal{Y}_z)\widehat{\otimes}_{\mathcal{O}(\mathcal{Y}_z)}\mathcal{O}(U'_i) \\
&= (\Pi^{N_0})'_b\widehat{\otimes}_{E[Y^+]}\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)\widehat{\otimes}_{\mathcal{O}(\mathcal{W}\times\mathbb{G}_m)}\mathcal{O}(\mathcal{Y}_z)\widehat{\otimes}_{\mathcal{O}(\mathcal{Y}_z)}\mathcal{O}(U'_i) \\
&= (\Pi^{N_0})'_b\widehat{\otimes}_{E[Y^+]}\mathcal{O}(U'_i) \\
&= \Big(\varprojlim_h V_h\Big)\widehat{\otimes}_{\mathcal{O}(\mathcal{W})[z]}\Big(\frac{\mathcal{O}(W_i)\{\{T,T^{-1}\}\}}{(Q(T))}\Big) && \text{(Recall } z\mapsto T^{-1}) \\
&= \varprojlim_{U_h\supset W_i}\Big(V_h\widehat{\otimes}_{A_h[z_h]}\Big(\frac{\mathcal{O}(W_i)\{\{z_h,z_h^{-1}\}\}}{(Q(z_h^{-1}))}\Big)\Big).
\end{aligned}$$

By [Buz07, Theorem 3.3], we can decompose each $V_h\widehat{\otimes}_{A_h}\mathcal{O}(W_i)$ into a direct sum $N_h \oplus F_h$ where N_h is a projective $\mathcal{O}(W_i)$-module of rank $\deg Q$, which is annihilated by $Q^*(z_h)$, and $Q^*(z_h)$ is invertible on F_i. (Recall $Q^*(T) := T^{\deg Q}Q(T^{-1})$ from [Buz07, §3].) Since z_h is invertible in $\mathcal{O}(W_i)\{\{z_h, z_h^{-1}\}\}$, we can simplify each term in the projective limit as follows:

$$V_h\widehat{\otimes}_{A_h[z_h]}\Big(\frac{\mathcal{O}(W_i)\{\{z_h,z_h^{-1}\}\}}{(Q(z_h^{-1}))}\Big) = V_h\widehat{\otimes}_{A_h[z_h]}\Big(\frac{\mathcal{O}(W_i)\{\{z_h,z_h^{-1}\}\}}{(z_h^{\deg Q}Q(z_h^{-1}))}\Big) = V_h\widehat{\otimes}_{A_h[z_h]}\Big(\frac{\mathcal{O}(W_i)\{\{z_h,z_h^{-1}\}\}}{(Q^*(z_h))}\Big).$$

Continuing with the calculation:

$$V_h\widehat{\otimes}_{A_h[z_h]}\Big(\frac{\mathcal{O}(W_i)\{\{z_h,z_h^{-1}\}\}}{(Q^*(z_h))}\Big) = V_h\widehat{\otimes}_{A_h[z_h]}\Big(\frac{\mathcal{O}(W_i)[z_h]}{(Q^*(z_h))}\Big) = \frac{N_h\oplus F_h}{Q^*(z_h)(N_h\oplus F_h)} = \frac{N_h\oplus F_h}{F_h} = N_h.$$

Therefore, each term in the projective limit is a projective $\mathcal{O}(W_i)$-module of rank $\deg Q$. Using the maps α_h and β_h as before, we can construct maps in the opposite direction from the transition maps given in the projective limit, and whose composition with the transition maps induce the identity, and hence the transition maps are all isomorphisms. Therefore, $\Gamma(U_i', \mathcal{N})$ is a projective limit of rank $\deg Q$ projective $\mathcal{O}(W_i)$-modules, whose transition maps are all isomorphisms, so $\Gamma(U_i', \mathcal{N})$ has the required property itself. $\square$

Proposition 5.17. *There exists an admissible cover $\{U_i\}_{i\in I}$ of $\mathcal{E}$, such that for each $i \in I$, there exists an open affinoid W_i of $\mathcal{W}$, such that κ restricts, on each irreducible component of U_i, to a finite surjective morphism onto W_i. Furthermore, $\mathcal{O}_{\mathcal{E}}(U_i)$ is isomorphic to an $\mathcal{O}_{\mathcal{W}}(W_i)$-algebra of endomorphisms of a finite projective $\mathcal{O}_{\mathcal{W}}(W_i)$-module.*
[BHS17, Lemma 3.10]

Proof. Fix an admissible affinoid cover $\{U_i'\}_{i\in I}$ of $\mathcal{Y}_z$ which we have constructed in the previous proposition. Set $U_i := f^{-1}(U_i')$ and $W_i := g(U_i')$. Then $\{U_i\}_{i\in I}$ is an admissible cover of $\mathcal{E}$. We will prove that each U_i is affinoid, and that $\mathcal{O}_{\mathcal{E}}(U_i)$ is isomorphic to a subalgebra of the $\mathcal{O}_{\mathcal{W}}(W_i)$-algebra $\mathrm{End}_{\mathcal{O}(W_i)}(\Gamma(U_i', \mathcal{N}))$.

If we can do this, then by [Che04, Lemma 6.2.10], this implies that each irreducible component of U_i map surjectively onto an irreducible component of W_i. We can choose a refinement of our cover so that W_i is irreducible, to complete the proof.

To proceed, recall that by [BHS17, Lemma 5.6], there is an isomorphism:

$$\mathcal{O}_{\mathcal{E}}(U_i) = \mathcal{O}(\mathcal{E})\widehat{\otimes}_{\mathcal{O}(\mathcal{Y}_z)}\mathcal{O}_{\mathcal{Y}_z}(U_i').$$

Let $M := \Gamma(\mathcal{E}, \mathcal{M}) = \Gamma(\mathcal{Y}_z, \mathcal{N}) = \Gamma(\mathcal{W}\times\mathbb{G}_m, \mathcal{N})$. Then by [BHS17, Lemma 5.5], there is an isomorphism of $\mathcal{O}_{\mathcal{Y}_z}(U_i')$-modules:

$$\Gamma(U_i, \mathcal{M}) = \Gamma(\mathcal{E}, \mathcal{M})\widehat{\otimes}_{\mathcal{O}(\mathcal{E})}\mathcal{O}_{\mathcal{E}}(U_i) = M\widehat{\otimes}_{\mathcal{O}(\mathcal{Y}_z)}\mathcal{O}_{\mathcal{Y}_z}(U_i') = \Gamma(U_i', \mathcal{N}).$$

The right hand side is a finite projective $\mathcal{O}(W_i)$-module by the previous proposition, and hence so is the left hand side. Recall we had seen that $\mathcal{M}$ can be viewed as a coherent sheaf on the product $\widehat{T}\times \mathrm{Sp}\,\mathcal{H}(K^p)^{\mathrm{sph}}$, where we note that the embedding $\mathcal{E}\hookrightarrow \widehat{T}\times\mathrm{Sp}\,\mathcal{H}(K^p)^{\mathrm{sph}}$ identifies $\mathcal{E}$ with the support of $\mathcal{M}$. This implies that the action of $\mathcal{O}_{\mathcal{E}}(U_i)$ on $\Gamma(U_i, \mathcal{M})$ is faithful, and this induces an injective map of $\mathcal{O}(W_i)$-algebras:

$$\mathcal{O}_{\mathcal{E}}(U_i)\hookrightarrow \mathrm{End}_{\mathcal{O}(W_i)}(\Gamma(U_i', \mathcal{N})).$$ $\square$

Proposition 5.18. *The space $\mathcal{E}$ is equidimensional, and has no embedded components. Furthermore, the morphism $f : \mathcal{E}\to\mathcal{Y}_z$ is finite, and the image under f of an irreducible component of $\mathcal{E}$ is an irreducible component of $\mathcal{Y}_z$.*
[BHS17, Proposition 3.11]

Proof. Let U be an element of the cover of $\mathcal{E}$ from the previous proposition. Let $V := \kappa(U)$. Then $B := \mathcal{O}_{\mathcal{E}}(U)$ is an $A := \mathcal{O}_{\mathcal{W}}(V)$-algebra acting faithfully on a projective A-module M of finite type. If y is a point of U with image x in V, then the ring $\widehat{B}_y$ is isomorphic to a sub-$\widehat{A}_x$-module of a free $\widehat{A}_x$-module of finite type. (We have used the fact that a projective module of finite type over a local ring is free.)

Let $\mathfrak{p}$ be any associated prime ideal of $\widehat{B}_y$. Then by definition there is an embedding $\widehat{B}_y/\mathfrak{p} \hookrightarrow \widehat{B}_y$ where we can view $\widehat{B}_y/\mathfrak{p}$ as an ideal of $\widehat{B}_y$. Since $\widehat{A}_x$ is an integral domain, and $\widehat{B}_y/\mathfrak{p}$ is a sub-$\widehat{A}_x$-module of a free $\widehat{A}_x$-module, it is torsion-free, and hence the Krull dimension of $\widehat{B}_y/\mathfrak{p}$ as an $\widehat{A}_x$-module is:

$$\dim_{\widehat{A}_x}(\widehat{B}_y/\mathfrak{p}) := \dim(\widehat{A}_x/\operatorname{Ann}_{\widehat{A}_x}(\widehat{B}_y/\mathfrak{p})) = \dim(\widehat{A}_x).$$

By [Gro67, Proposition 16.1.9], this upgrades to the statement:

$$\dim_{\widehat{B}_y}(\widehat{B}_y/\mathfrak{p}) = \dim(\widehat{B}_y/\operatorname{Ann}_{\widehat{B}_y}(\widehat{B}_y/\mathfrak{p})) = \dim(\widehat{A}_x).$$

But this is just the Krull dimension of $\widehat{B}_y/\mathfrak{p}$ as a ring. Notice that the quantity $\dim(\widehat{A}_x)$ does not depend on the choice of a point x in V, since V is irreducible, and hence A is an integral domain. In fact, this quantity is independent of the choice of V, since $\mathcal{W}$ is equidimensional. We may thus write $\dim(\widehat{A}_x) = \dim(\mathcal{W})$ for any point x in V.

Recall that the ring B is equidimensional if $\dim(B/\mathfrak{p})$ does not depend on the choice of a minimal prime ideal $\mathfrak{p}$. Since minimal prime ideals are associated, and we have shown that $\dim(\widehat{B}_y/\mathfrak{p}) = \dim(\mathcal{W})$ for any associated prime ideal $\mathfrak{p}$ and any point y of U, this implies the equidimensionality of B. Indeed, since $\dim(\widehat{B}_y/\mathfrak{p})$ does not depend on the choice of associated prime $\mathfrak{p}$ and point y of U, the only associated primes are minimal, i.e. isolated primes, and there can be no embedded primes for dimensionality reasons.

Refer to [BHS17, Proposition 3.11] for the remaining claims. □

Proposition 5.19. *The image of an irreducible component of $\mathcal{E}$ under $\kappa : \mathcal{E} \to \mathcal{W}$ is a Zariski-open subset of $\mathcal{W}$.*
[BHS17, Corollary 3.12]

Proof. Indeed, by the previous proposition, it suffices to prove that the image under g of an irreducible component of $\mathcal{Y}_z$ is Zariski-open in $\mathcal{W}$. The irreducible components of $\mathcal{Y}_z$ are themselves Fredholm hypersurfaces, since irreducible components of Fredholm hypersurfaces are themselves such, by [Con99, Theorem 4.2.2]. The claim then follows from the proof of [Che04, Corollary 6.4.4]. □

We have proved most of the Theorem 5.21 below.

Definition 5.20. Let $z \in \mathcal{E}(\overline{\mathbb{Q}}_p)$ be a $\overline{\mathbb{Q}}_p$-point of the eigenvariety, and let (χ, λ) be the image of z under the embedding $\mathcal{E}(\overline{\mathbb{Q}}_p) \hookrightarrow \widehat{T}(\overline{\mathbb{Q}}_p) \times (\operatorname{Sp}\mathcal{H}(K^p)^{\mathrm{sph}})(\overline{\mathbb{Q}}_p)$. The point z is said to be *classical* if there exists a non-zero T-equivariant map:

$$\chi \to J_B(\widetilde{H}^0(K^p)_{\mathrm{lalg}})[\lambda].$$

Let $\mathcal{E}(\overline{\mathbb{Q}}_p)_{\mathrm{cl}}$ denote the subset of classical points of $\mathcal{E}(\overline{\mathbb{Q}}_p)$.

Theorem 5.21.

(a) $\mathcal{E}$ is equidimensional of dimension equal to $\dim\mathcal{W}$. *Suppose $\mathcal{C}$ is an irreducible component of $\mathcal{E}$, then $\kappa(\mathcal{C})$ is a Zariski open subset of $\mathcal{W}$.*

(b) Let $z \in \mathcal{E}(\overline{\mathbb{Q}}_p)$ be a point, and let (χ, λ) denote the image of z under the embedding $\mathcal{E}(\overline{\mathbb{Q}}_p) \to \widehat{T}(\overline{\mathbb{Q}}_p) \times (\operatorname{Sp} \mathcal{H}(K^p)^{\mathrm{sph}})(\overline{\mathbb{Q}}_p)$. Suppose χ factors as $\chi = \chi_{\mathrm{alg}}\chi_{\mathrm{sm}}$ where χ_{sm} is smooth and χ_{alg} is a strictly dominant algebraic character. Then if χ has non-critical slope, then z is classical.

(c) The set $\mathcal{E}(\overline{\mathbb{Q}}_p)_{\mathrm{cl}}$ is Zariski dense in $\mathcal{E}(\overline{\mathbb{Q}}_p)$ and accumulates at every point of $\mathcal{E}(\overline{\mathbb{Q}}_p)_{\mathrm{cl}}$. The accumulation property means that each point of $\mathcal{E}(\overline{\mathbb{Q}}_p)_{\mathrm{cl}}$ admits a basis of affinoid neighbourhoods V such that $V \cap \mathcal{E}(\overline{\mathbb{Q}}_p)_{\mathrm{cl}}$ is Zariski dense in V.

[NT21, Proposition 2.22]

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